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ABSTRACT

This document consists of the two numbers of "The AMATYC Review" issued during publication year 1991-1992. The following articles are featured: (1) "Educational Reflections" (D. A. Crocker); (2) "Mathematics: An International View" (I. Malyshev, J. R. Becker, editors); (3) "Pandora's Rectangular Parallelepiped" (L. R. Tanner, editor); (4) "The Chalkboard" (J. Cain, J. Browne, editors); (5) "Problem Section" (M. W. Ecker, editor); (6) "Book Reviewss" (J. J. Edgell, Jr., editor); and (7) "Software Reviews", (S. Mah, editor). In addition, the fall 1991 issue contains: "Using the Calculator in a Prealgebra Course" (L. H. Boyd, V. M. Carson); "Supercurves: An Opportunity for Student Research in Precalculus Courses" (J. R. Smart); "On p-Secting an Angle" (S. P. Gordon); "A Useful Vector Decomposition Theorem" (A. E. Labarre, Jr.); "L'Hopital's Rule with Mathematica" (J. H. Mathews); "Compound Interest the e to the x Way" (B. Leonard, H. S. Shultz); "Name That Mathematician" (C. Ashbacher); "Mind's-Eye Proof" (R. L. Francis); "A Program for the Math Anxious at the University of Minnesota" (D. F. Robertson); and "Solving Word Problems Using the Foreign Language Algebra" (R. Pumford). Articles in the spring 1992 issue are: "A Time for Change in Remedial Mathematics" (E. D. Laughbaum); "An Application of Discrete Mathematics to Coding Theory" (L. J. Donohoe); "Symmetry in Irregular Convex Polygons" (T. W. Shilgalis); "Patterns Related to the Decimal Expansions of 2 to the negative nth" (A. Cusumano); "The Transformation of an Inconsistent Linear System into a Consistent System" (M. Paredes, M. Fatehi, R. Hinthorn); "Deriving the Area of a Circle" (J. D. Austin); "Hollywood Bowl Parking" (M. Bonsangue, H. S. Shultz); "An Inductive Proof of Fermat's Little Theorem" (J. Suzuki); "A Swiftly Convergent Sequence to Pi" (M. Schapira); "Poetry in Curve Sketching" (J. Z. Wiranowski); "Poetry in Mathematics (R. L. Francis); "The Arithmetic Connection" (P. W. Brunson); and "College Placement Testing: A Local Alternative" (W. L. Truman). (JMC)

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VOLUME 13, NUMBER 1 Fall 1991

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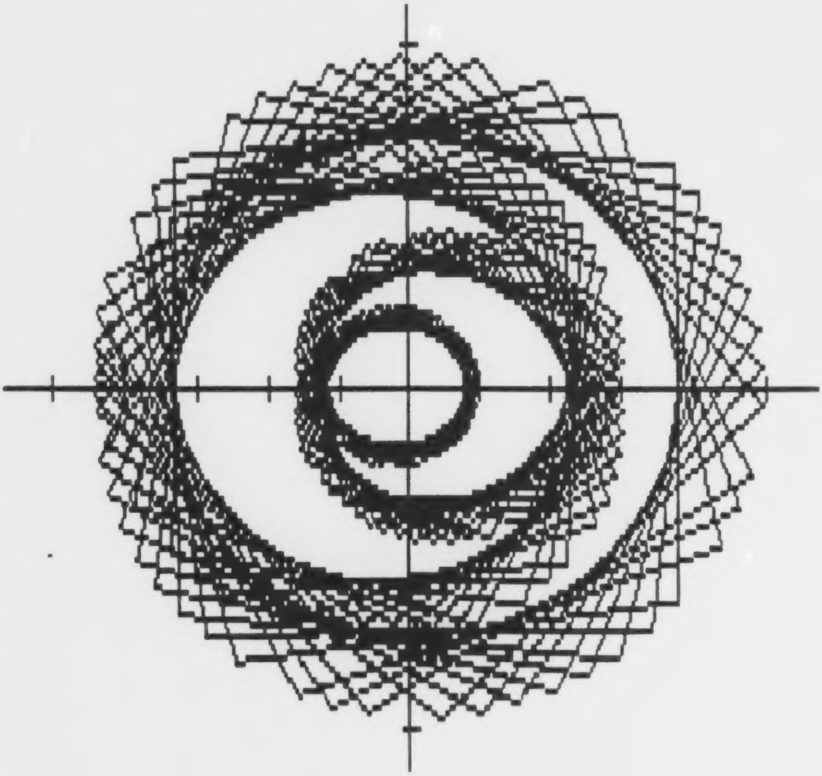

A CIRCULAR SAW
See John Jenkins' *Computer Art*

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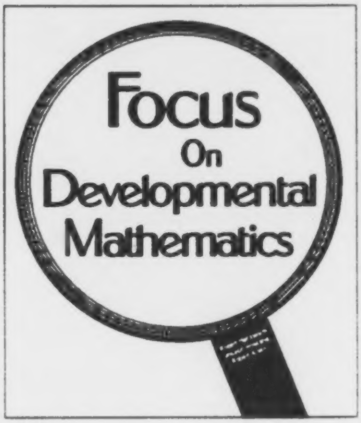

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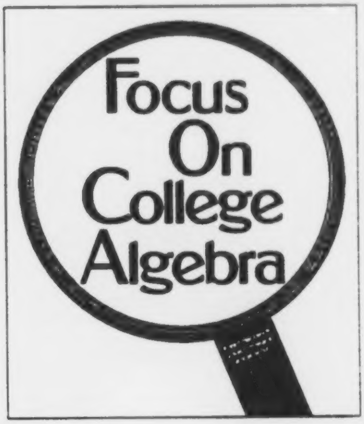

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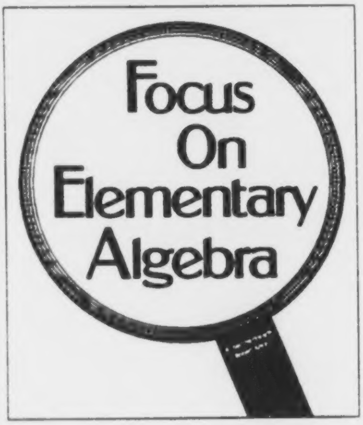

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- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

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Editor's Comments

On Being a Manuscript Reviewer

Most of the manuscripts that are submitted to *The Review* and eventually become published go through several rounds of review and revision. Our reviewers take the job of offering suggestions that will help authors improve their manuscripts very seriously. In that regard, reviewer Art Merifield (Senaca College, Toronto, Ontario, Canada) writes:

> I know there is a fine line between being thorough and simply discouraging an author. I know how personal and sensitive the creative process can be to some and I respect this. But I also know that the integrity of the entire (review) process depends upon accuracy. I sincerely hope that you will tell me if you feel that I am in any way harming the creative efforts of the authors whose papers I come in contact with. It has been my intention to be positive and supportive whenever possible—to encourage the most from an author without infringing upon his or her right of priority.

As an editor, I need to receive reviews that will help authors make their articles a valuable reading experience. Yes, reviewers should be "positive and supportive whenever possible." They should encourage the most from an author in a friendly, but frank, way.

And speaking of reviewers, readers will notice that I have not included a list of our current panel in this issue. From now on I plan on publishing that list only in our spring issue. This policy will leave more room to publish an increasing number of submissions.

Some Ideas for Articles

David Horowitz (Golden West College, Huntington Beach, CA) appended the following note to a recent review:

> The business calculus reform would certainly benefit from a critical re-evaluation (as is currently taking place in the "scientific" calculus reform projects), and *The AMATYC Review* can provide an excellent forum to initiate and direct this process.

I agree. I welcome manuscripts on the role of specialized calculus courses in a two-year college curriculum.

Sheldon Gordon (Suffolk County Community College, Selden, NY) writes:

> . . . one of the problems we face, and one that will become worse in the years ahead, is the aging nature of two-year (and other) faculty. It is bad enough that we have forgotten most of what we once knew; in addition, areas that we didn't know (that were on the fringes, so to speak, such as numerical analysis or statistics) have now gained major importance. Worse still, many new areas have developed that most of us know nothing about. Maybe you should consider an on-going faculty development series which will expound on some of these areas of mathematics.

While I do not plan on creating a special column or series, I would like our readers to consider the MATHEMATICAL EXPOSITION section of *The Review* as a good source for faculty enrichment articles. You are encouraged to share your expertise in emerging mathematical topics (that should be included in a two-year college curriculum) by submitting articles. Tell us what you are doing and how you are doing it.

Some Interesting Publications

I recently received the initial issue of *PRIMUS - Problems, Resources, and Issues in Mathematics Undergraduate Studies*. *PRIMUS* is a refereed quarterly that is described by its editor Brian J. Winkel as

> . . . a forum for exchanging problems, activities, and approaches which are interesting, applicable, real, engaging, and appropriate for undergraduate mathematics coursework.

PRIMUS is not a "Theorem-Proof" journal. Its main purpose is to offer teachers an opportunity to exchange ideas that will allow them to do a better job teaching mathematics. It is a journal that is most appropriate for two-year college educators. Write to the editor at Rose-Hulman Institute of Technology, Terre Haute, IN 47803 for subscription details.

I would also like to remind our readers about the *Recreational & Educational Computing Newsletter* edited by our problem section editor Michael W. Ecker. *REC* features programming, fun, and challenges. Write to the editor at 909 Violet Terrace, Clarks Summit, PA 18411 for subscription details.

Don Cohen, Editor

ABOUT THE COVER

Computer Art

by John H. Jenkins
Embry-Riddle Aeronautical University
Prescott, AZ 86301

John is Professor of Mathematics at ERAU. He received his BA from the University of Tennessee at Chattanooga and his MAT from the University of Florida. His current interest is in incorporating spreadsheet technology into the mathematics curriculum.

The computer generated image on the cover represents the winning entry in a computer art contest that I conducted in my Calculus II classes in the fall semester, 1990.

Master Grapher (Waits & Demana) is used for several assignments in Calculus II to explore polar, parametric, and conic graphs. A very precise graph can be obtained when the range is small and the number of points plotted (speed) is large. *Master Grapher* is a typical graphics package in which two points are plotted and a line joining them is then drawn. The fewer points that are plotted, the choppier the graph appears. This can be carried to extreme by making the range large and the number of points plotted small in comparison. When this is done in polar or parametric mode, computer designs are generated.

For the past two semesters, I have conducted a computer art contest in my Calculus II classes. As an extra credit assignment, my students can turn in their best original creation. The students must furnish the range and speed used so the projects can be duplicated. About half of the students participate. They seem to enjoy the assignment. The students receive four extra credit points added to their lowest test score for entering. I then do a preliminary judging and pick the eight or ten that I find most "eye appealing." I then run off all of the semifinal designs and hang them on the wall in the classroom. The students then vote on the design they like the best. The winning entry is then framed and hung on my office wall.

I give out a handout on some designs for the students to try so they get an idea of how the process works. (A copy of this handout is available from the author.) I also demonstrate a computer art design in class as we're working on polar graphing. The students are on their own to experiment with different functions. I've been doing this for two semesters and have received sixty-two entries. No two of these have been remotely similar.

Here are some functions and settings for you to try.

1. Cover Design - Circular Saw by Terry Holoun (Winning Entry - Fall 1990)
   $P(t) = 3 + 2{*}\sin(t/4)$
   t-Range: 0 - 999,999 Speed: 200 Viewing Window $[-6, 6, -6, 6]$
   (By varying t slightly, you get a much different design. Try 0 - 999,000.)

2. Pick Up Sticks by Monte Byrom (Second Place - Fall 1990)
   $P(t) = \mathrm{sqr}(9{*}\sin(7{*}t/3))$
   t-Range: 0-3500 Speed: 400 Viewing Window $[-4, 4, -4, 4]$
   (You need to watch this one being graphed! Watch what happens when you change the speed to 200.)

   The same effect can be achieved using the Parametric Grapher.

3. Untitled by Joe Hill (Winning Entry - Spring 1991)
   $X(t) = 5{*}\cos(t) - 2{*}\cos(2.5{*}t)$
   $Y(t) = 5{*}\sin(t) - 2{*}\sin(2.5{*}t)$
   t-Range: 0-421 Speed: 100 Viewing Window $[-10, 10, -10, 10]$

4. The Butterfly Curve (Fay, 1989)
   $P(t) = \exp(\cos(t)) - 2{*}\cos(4{*}t) + (\sin(t/12))\hat{}\,5$
   t-Range: 0-65 Speed: 1000 Viewing Window: $[-3, 5, -5, 5]$

References

Fraleigh, P. (1990). *Exploring Calculus with the IBM PC Version 2.0*. Reading, MA: Addison-Wesley Publishing Co.

Fay, T. (1989). The Butterfly Curve. *The American Mathematical Monthly*, 96, 442-443.

Waits, B. & Demana, F. (1990) *Master Grapher*. Reading, MA: Addison-Wesley Publishing Company, Inc.

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James W. Hall, Parkland College

ELEMENTARY ALGEBRA FOR COLLEGE STUDENTS, Fourth Edition

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James W. Hall, Parkland College

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Trigonometry

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New for 1992

Calculus

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CALCULUS, Fifth Edition *New for 1991*

CALCULUS OF A SINGLE VARIABLE
Earl W. Swokowski, Marquette University

CALCULUS, Third Edition
Dennis G. Zill, Loyola Marymount University

CALCULUS, VOLUMES I and II
Tom Dick, Oregon State University (Calculus Curriculum Project)
Charles Patton, The Hewlett Packard Corporation

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VIEWPOINT

Using the Calculator in a Prealgebra Course

by Linda Boyd and Virginia Carson
DeKalb College
Clarkston, GA 30021

Linda is Associate Professor of Mathematics at DeKalb College. Her chief interests are in the area of testing and the use of technology in instruction.

Virginia is Assistant Professor of Mathematics at DeKalb College. She is a PhD student in Mathematics Education at Georgia State University.

The hand-held calculator is an accepted computational tool in virtually every field of endeavor. In secondary and technical schools, in two- and four-year colleges, and in universities, calculators are widely used for problem solving in almost every course that requires numeric computations. An exception to this practice is found in many college developmental mathematics programs.

National Recommendations

Why this has occurred is difficult to explain, particularly in light of the recommendations made by national mathematics organizations and commissions. For example, the National Council of Teachers of Mathematics (1980) recommended that "mathematics programs take full advantage of the power of calculators and computers at all grade levels" (p. 1). In *Everybody Counts: A Report to the Nation on the Future of Mathematics Education*, commissioned by the National Research Council (1989), Lynn Arthur Steen stated: "The teaching of mathematics is shifting from preoccupation with inculcating routine skills to developing broad-based mathematical power.... The repertoire of skills which now undergird mathematical power includes not only some traditional paper-and-pencil skills, but also many broader and more powerful capabilities" (p. 82). Steen further claimed that students should be able to estimate, decide between reporting exact answers and approximations, select appropriate mathematical operations, and use calculators correctly.

Among the underlying assumptions for the study of mathematics in Grades 9-12 found in the Commission on Standards for School Mathematics report *Curriculum and Evaluation Standards for School Mathematics* (1989, p.124) are:

- The level of computational proficiency suggested in the K-8 standards will be expected of all students; however, no student will be denied access to the study of mathematics in grades 9-12 because of a lack of computational facility.
- Scientific calculators with graphing capabilities will be available to all students at all times.

Reporting for the National Science Board Commission on Precollege Education in Mathematics, Science and Technology (1984) in *Educating Americans for the 21st*

Century, Henry Pollak stated, "In the past such recognition skills and conceptual understanding have been learned as a by-product of manipulative drill, if learned at all. The challenge now is to teach skills and understanding even better while using the power of machines to avoid large time allotments to tedious drill. Some blocks of traditional drill can surely be curtailed" (p. 9).

Prevalence of Arithmetic Courses in Developmental Mathematics Programs

In light of these recommendations it would seem college developmental mathematics programs should begin with the study of algebra, and each student should have access to a scientific calculator. But this seems not to be the case. The American Mathematical Association of Two-Year Colleges and the City University of New York distributed a questionnaire on the evaluation of remedial mathematics programs to all mathematics chairpersons of institutions of higher education (Akst, 1985). It was found that of the responding institutions, 76% offered arithmetic while 87% offered elementary algebra and 52% intermediate algebra. In this survey, the program goal most commonly reported was to help students prepare for subsequent mathematics courses. If preparation for subsequent mathematics courses is the goal, why is so much arithmetic being taught? Should we not begin with the study of algebra and require that students become proficient in the use of the calculator? There was no question on the survey which addressed calculator usage. This indicates that the reality of mathematics education prior to 1985 was quite different from present recommendations.

The most recent survey of undergraduate programs in the mathematical and computer sciences conducted by the Conference Board of the Mathematical Sciences found similar results (Albers, 1987). Of 479,000 students enrolled in remedial mathematics courses at four-year colleges and universities, 46,000 (10%) were enrolled in arithmetic or general mathematics; of 482,000 students enrolled in remedial mathematics at two-year colleges, 142,000 (29%) were enrolled in arithmetic or general mathematics. Though it was reported that at two-year colleges the impact of hand-held calculators was substantially greater than that of computers, no remedial course was among the 13 courses with significant calculator usage. This seems to indicate that many of these remedial courses are taught with the traditional emphasis on paper-and-pencil drill.

Arithmetic Computations as Basic Skills

Why are so many of the courses offered in developmental mathematics still arithmetic or general mathematics, and why do they stress primarily paper-and-pencil skills? One explanation for the lack of incorporation of the calculator into these courses is the deeply held belief of many teachers that until basic skills are mastered, it is impossible for the student to progress to college-level mathematics. For many of these teachers, basic skills mean arithmetic computations and little else.

Students who take prealgebra in college are not enrolling in new mathematics; they have studied arithmetic through the elementary school years and some of them through the high school years as well. The fact that these students do not qualify for college mathematics testifies that they were not successful with previous experiences in drill and practice.

Anderson (1983) examined the changing nature of the use of arithmetic and declared paper-and-pencil algorithms obsolete. He attributed more importance in modern mathematics to decimals and decimal approximations. In addition, he recommended that students need to understand and estimate percents rather than compute using formal algorithms. He suggested that the availability of calculators and computers has affected what students should know about arithmetic.

Albers (1983) took issue with recommendations that students should master basic mathematics skills before the use of a calculator is allowed. He proposed that mathematics would be more interesting to adult students if data analysis using the calculator were incorporated. He speculated that solving application problems using technology may help develop organizational skills.

Hilton (1981) said "with cheap hand-held calculators available it is manifestly absurd to be concentrating on those dreary mechanical calculations" (p. 78). Usiskin (1985) reported the following example of the emphasis on skill rather than understanding. In the 1981-82 National Assessment of Educational Progress results, 57% of 13-year-olds and 72% of 17-year-olds correctly calculated the exact answer to a multiple-choice multiplication problem that asked for the product of a two-place decimal and a one-place decimal. Later the same students were asked to estimate the results of a similar product:

ESTIMATE the answer to
3.04 X 5.3.
(a) 1.6
(b) 16
(c) 160
(d) 1600
(e) I don't know

This estimation item was correctly answered by only 20.7% of 13-year-olds (about what would be expected from random guessing) and only 36.6% of 17-year-olds. These and other related results stem from the pedagogical emphasis on algorithms rather than reasoning and problem solving.

Calculator and Testing Issues

One explanation for the emphasis on paper-and-pencil algorithms is the prevalence of skill-based items on tests. Heid (1988) stated that if teachers believe that their students must demonstrate algorithmic calculations on standardized tests, then teachers will plan their instruction accordingly. She further stated that students believe "that the test is the major determinant of what is important in mathematics" (p. 711).

The Mathematical Association of America through its Committee on Testing (COT) is developing a series of mathematics placement examinations that require students to use calculators for some of the items. L. Boyd, as a member of COT, served on the test panel to develop the Calculator Arithmetic and Skills Test (Coping with

Calculators, 1988). Other testing groups are also developing tests which allow or require calculator use. On Saturday, June 1, 1991, the College Board administered its first Achievement Test requiring the use of a scientific calculator: the Mathematics Level IIC Achievement Test. The 1989 report of the College Board/Mathematical Association of America Committee on Mutual Concerns included a statement of support for the concept of the use of calculators on the AP Calculus Examination in 1992 (Armstrong, 1989). On the new SAT-I, to be introduced in 1993-94, students will be permitted to use calculators. The roll of the calculator in the testing environment is rapidly being redefined.

Research With Calculators in Prealgebra

With these calculator and curriculum precepts in mind, the authors initiated a study of the use of the calculator to introduce concepts of algebra in a prealgebra course. The purpose of the study was to implement recommendations on calculator use and to evaluate consequent student performance. It was hoped that given a course with standard content and test items the time required to teach calculator skills would not detrimentally affect test outcomes.

Methodology: Two sections of prealgebra at DeKalb College were selected. DeKalb is a multi-campus, two-year institution in metropolitan Atlanta with approximately 13,000 commuting students. The mean student age is 24.7 years. The experimental section was taught by the authors as a team. The 38 students in the experimental group were asked to bring a calculator to class daily. These students were encouraged to use their calculators in class, on homework, and on tests. Instruction in calculator use was provided as necessary. The control section was taught by Boyd alone. The 28 students in this section were not given any instruction in calculator use and were not allowed to use a calculator on any test.

Several pedagogical criteria were adopted for the experimental section. Throughout the quarter the instructors emphasized the difference in exact and approximate results, appropriate use of the calculator in problem solving, the role of estimation in analyzing an arithmetic calculation, and the reasonableness of answers. This use of the calculator led naturally to an emphasis on problem solving in the course.

As the lowest entry level mathematics course, prealgebra was designed to contain 40% arithmetic and 60% algebra. The arithmetic content was operations and applications of fractions, decimals, and percentages, and definition of square roots. The instructors wrote supplementary materials that featured applications that fit real-life situations. For example, compute the cost of 1.95 pounds of ground beef at $1.69 per pound. This type of problem provided a vehicle for stressing estimation, approximation, and reasonableness of the answer.

Other problems on these handouts required students in the experimental group to compare the arithmetic order of operations to the standard for their calculators. The students were also asked to evaluate the powers of decimal numerals rather than the powers of integers. The students in the control group performed these operations by hand. Some problems purposely involved calculations with results that appeared in scientific notation in the calculator display. Again, the students in the experimental group had to decide how to use their calculators. A large number of the arithmetic expressions were described in words, for example, the square of the sum of 3.56 and 8.91. Many of the problems required multiple steps in the solution processes.

The supplementary materials were used both in the experimental group and in the control group. The same calendar of homework assignments was used in both classes. The unit tests were designed to parallel those given in the regular sections of the prealgebra course and were identical for both sections. The main difference between the two sections was that students in the experimental section were taught to use the calculator and were encouraged to use the calculator in class, on homework, and on tests. The students in the control section were not taught to use the calculator and were not allowed to use the calculator on tests.

The discussions revolving around the calculator in the experimental class allowed the students to be active learners. In this laboratory-class environment small group discussion and peer teaching were permitted. In particular, the lesson on scientific notation generated excitement in the classroom as students compared their different calculator displays and struggled to reconcile the outcomes.

Results: Data was collected for the SAT math scores, SAT verbal scores, and scores on the Basic Skills Exam (BSE). These tests do not permit calculator use. Also, during the quarter, data was accumulated for scores on each of five unit tests and scores on the final exam. The BSE is a placement test used by the state, and the final exam is departmental, comprehensive, and multiple choice. The means for each group are displayed in the following table.

Mean Scores

	Experimental Group	Control Group
SAT (math)	277	294
SAT (verbal)	270	287
BSE (math)	57	58
BSE (reading)	63	67
BSE (English)	59	62
Test 1	68	62
Test 2	74	74
Test 3	72	63
Test 4	56	45
Test 5	41	38
Final Exam	71	67

Though the entering aptitudes and skills as measured by the SAT and BSE scores were lower for the experimental group than for the control group, the course test results were the same (Test 2) or higher for the experimental group than for the control group. In order to determine whether these results were statistically significant an analysis was made using a *t*-test where the distributions were approximately normal (SAT, Test 3, Test 4, Test 5, Final Exam) and a median test where the distributions were not normal (BSE, Test 1, Test 2). No significant difference existed on any of the tests at the .05 level.

Conclusions: The experimental group performed as well as, if not better than, the control group in the prealgebra course. Since the evaluation instruments did not contain any calculator-active items, there was no measure of additional calculator skills gained by the experimental group. They certainly gained the traditional skills taught in the course at the same or higher level than the control group and, in addition, were taught to use a scientific calculator. The time spent in class teaching calculator skills did not detrimentally affect the mastery of the objectives contained in the unit tests and final examination.

Despite the fact that the control group was not allowed to use the calculator on tests, 86% of the students in this group reported in a survey administered at the end of the quarter that they did use a calculator on homework assignments. It seems that these students are trying to tell us something. They protested that in-class use of calculators had not been permitted (91%) and were generally negative in their attitudes toward the course. This was in marked contrast to the positive attitude and excitement of the students in the experimental group. Hembree and Dessart found similar results for students in Grades K-12. They reported, "Students using calculators possess a better attitude toward mathematics and an especially better self-concept in mathematics than noncalculator students. This statement applies across all grades and ability levels" (1986, p. 96).

Summary: The calculator was used to introduce concepts of algebra in a prealgebra course. Students using the calculator gained a solid foundation for the study of college mathematics. The emphasis on algebra over arithmetic and the added opportunities for problem solving are in line with recommendations for mathematics curriculum. The results of this study indicate that introducing the calculator is a viable alternative for the prealgebra course. Given the fact that students are going to use calculators anyway, we should take advantage of these tools to enrich and enliven our courses.

Conclusion

Mathematics educators today have an opportunity to review and revise curriculum in light of the availability of low-cost calculators. Such curriculum review should question longstanding precepts of mathematical content. Preparing students for undergraduate mathematics courses requires incorporating problem-solving strategies and less dependence on rote performance. Students not trained in this fashion will not be able to meet the demands of either academics or industry. In addition, students realize that many of the problems in their mathematics textbooks are irrelevant and boring. The potential for excitement generated by technology-based curriculum changes can transform the classroom into a laboratory for active learning.

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MATHEMATICAL EXPOSITION

Supercurves: An Opportunity for Student Research in Precalculus Courses

by James R. Smart
San Jose State University
San Jose, CA 95192

SJSU

Jim received his PhD from George Peabody College of Vanderbilt University. He was designated the "Outstanding Professor" at SJSU in 1983-84. He is currently the director of the Center for Mathematics and Computer Science Education at SJSU.

Several years ago in a precalculus class, I referred to the curve with the equation $x^4 + y^4 = 1$ as a **supercircle**. We sketched its graph on the same axes as the graph of the associated circle $x^2 + y^2 = 1$, as shown in Figure 1. The students seemed interested, and then one student asked, "Are there other curves like supercircles?" From that fortunate question came elementary research by students on some of the ideas in this paper. Since that time, I often include supercircles and curves like supercircles in classes at the precalculus level (and sometimes above).

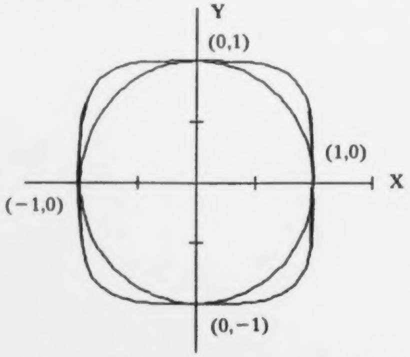

Figure 1. Circle $x^2 + y^2 = 1$ and associated supercircle $x^4 + y^4 = 1$.

"Curves like a supercircle" could have various interpretations, and the research would proceed accordingly. The interpretation here is the new equation is determined by replacing the variables x and y by x^2 and y^2, respectively, as indicated by the notation $(x^2 + y^2 = 1) \rightarrow (x^4 + y^4 = 1)$. We can name these new curves **supercurves**.

Some of the many possible questions that your students may consider in order to generalize the notion of supercircle follow. They are presented in an order appropriate for use in a classroom. Answers, partial answers, or suggestions are included after each question.

Question 1. What are some specific examples of equations for curves and equations for their associated supercurves?

$(y = ax^2) \rightarrow (y^2 = ax^4)$, for $a > 0$	superparabola
$(ax+by+c=0) \rightarrow (ax^2+by^2+c=0)$	superline
$(y=ax^3) \rightarrow (y^2=ax^6)$, for $a > 0$	supercubic

The graphs of the parabola $y=4x^2$ and the associated superparabola $y^2=4x^4$ are shown in Figure 2a. The graph of the line $x+y = 4$ and the associated superline $x^2+y^2=4$ are shown in Figure 2b.

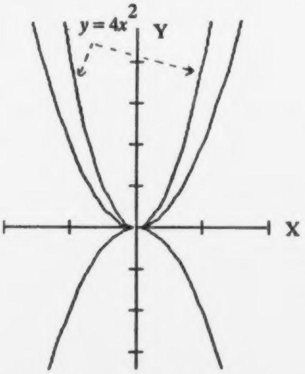

Figure 2a. Parabola $y=4x^2$ and associated superparabola $y^2=4x^4$.

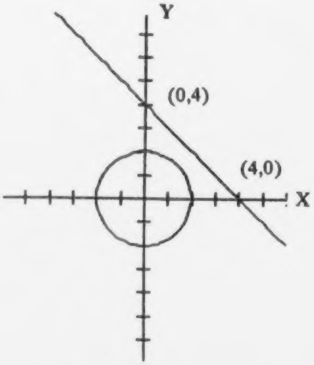

Figure 2b. Line $x+y=4$ and associated superline $x^2+y^2=4$.

Question 2. Not all the points of the supercircle $x^4 + y^4 = 1$ are the same distance from the origin. The minimum distance is one. What is the maximum distance?

Because of symmetry we can consider only the first quadrant. At the precalculus level, we also can assume the maximum distance lies along the line $x = y$. In one class, a student suggestion to use polar coordinates worked well. Since the equations connecting rectangular and polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$, the equation of the given supercircle is $r^4\cos^4\theta + r^4\sin^4\theta = 1$. We can solve this equation for r when the angle is 45 degrees and $\sin\theta = \cos\theta = 1/\sqrt{2}$ as follows:

$$r^4\cos^4\theta + r^4\sin^4\theta = r^4\left(\frac{1}{4}\right) + r^4\left(\frac{1}{4}\right) = 1 \Rightarrow r = 2^{1/4}, r > 0$$

Thus the maximum distance from the origin to a point on the given supercircle is $2^{1/4} \approx 1.1892$.

Question 3. The equation for a superline is the equation for a conic. Can you find examples of original equations such that the superline is a degenerate conic? Such that the superline has no graph?

If the original equation is $y = mx$, $m > 0$, the superline is of the form $y^2 = mx^2$. Then

$$y^2 - mx^2 = 0, \text{ and } (y - \sqrt{m}x)(y + \sqrt{m}x) = 0.$$

The graph is a pair of intersecting lines through the origin. If the original equation is $x = 5$, the equation for the superline is $x^2 = 5$, which is a pair of vertical lines. The original equation $x + y + 5 = 0$ is an example in which the superline $x^2 + y^2 + 5 = 0$ has no real points on its graph.

Students should be encouraged to try to find other examples in order to discover a generalization about points and their images with supercurves before studying the answer to the next question.

Question 4. How is the process of replacing x by x^2 and y by y^2 to find the equation of the supercurve related to the concept of points on the first curve and their images on the supercurve? How is it related to the concept of transformations or mappings?

Students should recall or learn that a mapping of one set into another is a pairing of elements of the first set with a subset of the second so that each element of the first is paired with exactly one element of the subset of the second, and each element of the subset of the second is paired with at least one element of the first. If the subset is equal to the second set, then the mapping is onto. The mapping is one-to-one if each element of the subset of the second set is the image of exactly one element of the first. A transformation is a one-to-one onto mapping.

To investigate what happens to individual points in the correspondence of curve to supercurve, we need to write equations for the correspondence between the original variables x and y and new variables x' and y' for the supercurve. Since $x = x'^2$ and $y = y'^2$, we have $x' = \pm\sqrt{x}, y' = \pm\sqrt{y}$ for the desired set of equations. When we see the equations in this form, however, we also realize we must put restrictions on the values for x and y: $x \geq 0$ and $y \geq 0$.

If we take a specific point, say (4,9), then the images are (2,3), (−2,3), (2,−3), and (−2,−3). The points A(1,1), B(1/4, 1/4), and C(4,4) and their images help reveal a pattern, as indicated in Figure 3.

We have seen that a problem arises with negative coordinates for x or y, because of the square roots. Consider the images of (−4,−9). This point has no real image points. Now students can begin to realize that a point must have nonnegative coordinates in order to have an image. If we are given a set of points on a curve, then only those points in the first quadrant have images on the associated supercurve. Points in the interior of the first quadrant have four images. This idea is very helpful in sketching the graph of a supercurve.

We can conclude that the correspondence between a curve and the associated supercurve is neither a transformation nor a mapping as described here, but something new. We can consider it a one-to-many mapping from the xy-plane to the $x'y'$-plane with equations

$$x' = \pm\sqrt{x},\ y' = \pm\sqrt{y},\ x \geq 0 \text{ and } y \geq 0. \tag{1}$$

Each element of the first set is paired with at least one element of the second set.

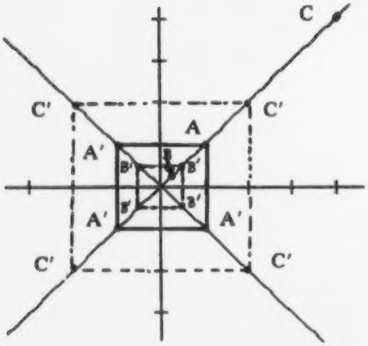

Figure 3. Graph of A(1,1), B(1/4,1/4),C(4,4) and their images A′ (±1, ±1), B′ (±1/2, ±1/2), C′(±2, ±2).

Question 5. Which points will be common to a curve and the associated supercurve?

If the curve and supercurve are graphed using the same set of axes, any possible answer will be restricted to points in the first quadrant. Setting $x = \pm\sqrt{x}$ and $y = \pm\sqrt{y}$ in the equations (1) shows that x and y must be zero or one. Thus the only common points can be (0,0), (1,1), (0,1), and (1,0). These can be considered invariant points in the sense that if they are on the original curve, at least one of their images will be on the supercurve.

Question 6. Are there connections between the concept of supercurve and the concept of symmetry?

A supercurve with real points will be symmetric with respect to the origin if the original curve has positive integral powers of x and y. The question of whether symmetry can be considered an invariant property is more difficult to answer. If we think of the original equation (not just the points in the first quadrant), then replacing the variables by the squares of the variables will not cause the symmetry with respect to the origin or the axes to be lost, as long as the supercurve has real points. From this viewpoint, a curve such as $x + y = 1$, which has no symmetry, is associated with the supercurve $x^2 + y^2 = 1$ that has various symmetries. The parabola $y = x^2$ is symmetric with respect to the y-axis, while the associated superparabola $y^2 = x^4$ retains this symmetry and is symmetric with respect to the x-axis and the origin as well.

Question 7. What can we learn about starting with the supercurve and finding the original curve connected with it?

Assuming that any equation can be considered the equation for a supercurve, then we might decide that the original curve, which can be called the **subcurve**, can be found by replacing x^2 by x and y^2 by y. Thus we might have

$(x^2 + y^2 = 1) \rightarrow (x + y = 1)$, a subcircle
$(x^4 + y^4 = 1) \rightarrow (x^2 + y^2 = 1)$, a subsupercircle.

There are some fairly obvious complications. For example, for a parabola such as $y = ax^2$, considered as a supercurve, we need to decide whether to use $\sqrt{y} = ax$ or

$-\sqrt{y} = ax$ or both as the subparabola. In general, up to four points on the supercurve are mapped into the same first quadrant point on the subcurve.

Question 8. The equation $x^4 + y^4 = 1$ has been described as a supercircle. Then $x^8 + y^8 = 1$, which results from replacing x and y in the equation of the supercircle by x^2 and y^2, can be called a **supersupercircle.** What can we learn about supersupercurves by answering questions similar to those about supercurves?

For example, we could follow a similar sequence of steps as in question 2 to find the maximum distance of the supersupercircle from the origin. We could also investigate the potential limiting nature of the supercurve-forming process by plotting

$$(x^2+y^2=1) \rightarrow (x^4+y^4=1) \rightarrow (x^8+y^8=1) \rightarrow \ldots$$

as shown in Figure 4. A student might say the curves become "more and more convex" and take on a shape closer to a square than a circle.

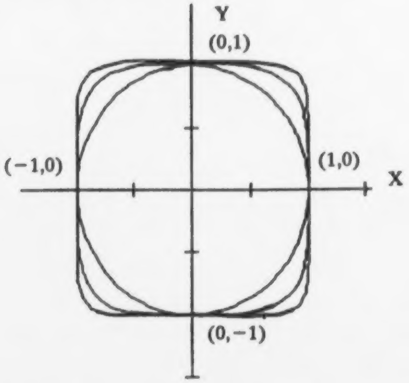

Figure 4. Potential limiting nature of supercurve process $(x^2+y^2=1) \rightarrow (x^4+y^4=1) \rightarrow (x^8+y^8=1) \rightarrow \ldots$

Question 9. What can we learn about the process of defining curves analogous to supercurves by replacing variables in an equation by some power other than two?

For example, we can replace x by x^3 and y by y^3 and consider questions similar to those in this article.

Among the possible benefits to students who investigate questions about the process of finding supercurves are the following:

- They encounter mappings that are not one-to-one and hence are uncommon at the precalculus level.
- They participate in research on a nontrivial topic.
- They gain experience in graphing.
- They may become interested in using a computer algebra system or a graphing utility as a practical way of solving graphing problems.
- They may enjoy an enrichment activity that is open-ended and less structured than textbook exercises.

Students who have a chance to begin research and exploratory-type activities very early in their mathematical studies are exposed to aspects of the subject that can lead them into patterns of creative thought that will be of benefit as they progress to higher mathematics.

For Further Study:

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On *p*-Secting an Angle

by Sheldon P. Gordon
Suffolk Community College
Selden, NY 11784

Shelly is professor of mathematics at Suffolk Community College. He is currently working on the Harvard Calculus Reform project and is particularly interested in applications of computers to all aspects of undergraduate mathematics.

One of the most famous "facts" in mathematics is Galois' result on the impossibility of trisecting an angle using only a straight edge and compass construction. The question of whether an arbitrary angle θ can be subdivided into n equal parts, however, is not widely considered. Yet, it is a natural and interesting extension. This article considers this question and then examines some interesting numerical patterns that arise in conjunction with answering the initial question.

p-Secting an Angle

Suppose that n is any multiple of 3. It is clearly impossible to subdivide an angle θ into n equal parts. For, if we could do so, then we need merely triple the resulting angle and so have performed a trisection.

On the other hand, it is obvious that we can subdivide an angle into n equal parts if n is a power of 2. By extending these two arguments, we see that the only cases which bear consideration are situations where n is a prime number, $p \geq 5$. If an angle θ can be p-sected for some prime p, then it can also be n-sected for any power of p, $n = p^k$. Similarly, if an angle can be p-sected for a variety of different primes, $p_1, p_2, \ldots$, then it can be p-sected for any composite number composed of products of powers of these primes.

To see how to proceed, let's consider the standard approach to proving that an angle cannot be trisected (see Birkhoff & MacLaine, 1977). If the angle θ were to be trisected, then we would be able to produce an angle Φ such that $\theta = 3\Phi$, and so,

$$\cos\theta = \cos 3\Phi = 4\cos^3\Phi - 3\cos\Phi\,. \tag{1}$$

This result can be obtained either by using the obvious trigonometric manipulation based on $\cos(2\Phi + \Phi)$ or by using the less familiar identity

$$\cos(A + B) + \cos(A - B) = 2\cos A\cos B,$$

where we set $A = (n-1)\Phi$ and $B = \Phi$ to obtain

$$\cos n\Phi = 2\cos(n-1)\Phi\,\cos\Phi - \cos(n-2)\Phi\,. \tag{2}$$

The underlying problem of trisecting the angle θ using just a straight edge and compass construction involves applying a pair of tools that are, respectively, linear and quadratic forms. Galois' theory assures us that this can be achieved in a finite number of steps if the corresponding equation above is reducible over an extension of degree 2 of the rationals. For instance, if $\theta = 270°$, then

$$4\cos^3\Phi - 3\cos\Phi = \cos 270° = 0,$$

so that

$$\cos\Phi\,(4\cos^2\Phi - 3) = 0.$$

This leads to a rational solution with $\Phi = 90°$ as well as the irrational solutions $\cos\Phi = \pm\sqrt{3}/2$. However, for an arbitrary angle θ, equation (1) is *irreducible* over the field of rationals. The term on the right, $\cos\theta$, can be any value between -1 and 1 and the resulting equation will certainly not have rational roots for every possible case. This implies that the equation is also irreducible over any quadratic extension of the rationals. Thus, in general, an angle cannot be trisected.

Suppose now that we consider an analogous argument with $p = 5$. We therefore consider $\theta = 5\Phi$ and attempt to write

$$\cos 5\Phi = \cos\theta.$$

After some trigonometric manipulations using either the obvious identities or the result in (2), we find that

$$\cos 5\Phi = 16\cos^5\Phi - 20\cos^3\Phi + 5\cos\Phi = \cos\theta, \tag{3}$$

or equivalently,

$$16u^5 - 20u^3 + 5u = b.$$

This equation is clearly irreducible over the field of rationals for any arbitrary choice of $b = \cos\theta$ since we know that an arbitrary polynomial of degree $n > 4$ is irreducible. Thus, we see that this equation cannot be solved in closed form for arbitrary angle Φ. Therefore, an arbitrary angle θ cannot be 5-sected.

By way of comparison, we consider the equivalent case with $n = 4$ where we obtain

$$\cos 4\Phi = 2\cos^2 2\Phi - 1 = \cos\theta = b.$$

Here,

$$\cos^2 2\Phi = \frac{b+1}{2}.$$

Hence, $\cos 2\Phi$ can be found in closed form since $b + 1 = \cos\theta + 1 \geq 0$. In turn, we can use this expression to determine $\cos\Phi$ for any angle θ.

On the other hand, for any prime p other than 2, as we saw with $p = 3$ and $p = 5$, it is evident that the resulting polynomial in $\cos\Phi$ will not be reducible over a quadratic extension of the rationals for arbitrary angle θ. Therefore, the corresponding angle cannot be p-sected. Thus, the only equal subdivisions of an arbitrary angle that can be performed are those corresponding to the powers of 2.

We refer the interested reader to a related article (Gleason, 1988) involving the geometric constructions that become possible provided that angle trisection is allowable.

Some Surprising Patterns

The polynomial equations in $\cos \Phi$ that arise for different values of n are of interest themselves. There are fascinating patterns relating the coefficients in these equations, and we will indicate some of them here. However, we will leave it to the interested readers and their students to derive these expressions and to determine other possible relationships. (In fact, the search for such patterns provides a nice motivation for students to practice the use of some important trigonometric identities in a trigonometry or elementary functions course.)

To simplify $\cos^n\Phi$ for specific values of n, we will follow the convention used earlier and write $u = \cos \Phi$. Therefore,

$n = 2$: $2u^2 - 1$
$n = 3$: $4u^3 - 3u$
$n = 4$: $8u^4 - 8u^2 + 1$
$n = 5$: $16u^5 - 20u^3 + 5u$
$n = 6$: $32u^6 - 48u^4 + 18u^2 - 1$
$n = 7$: $64u^7 - 112u^3 + 56u^3 - 7u$
$n = 8$: $128u^8 - 256u^6 + 160u^4 - 32u^2 + 1$
$n = 9$: $256u^9 - 576u^7 + 432u^5 - 120u^3 + 9u$
$n = 10$: $512u^{10} - 1280u^8 + 1120u^6 - 400u^4 + 50u^2 - 1$

In general, we can write

$$P_n(u) = A_{n,0}\,u^n + A_{n,1}\,u^{n-1} + A_{n,2}\,u^{n-2} + \dots + A_{n,n-1}\,u + A_{n,n}$$

It is immediately evident that:

1. $A_{n,0} = 2^{n-1}$ for all n;
2. $A_{n,1} = A_{n,3} = A_{n,5} = \dots = 0$ for all n;
3. The signs of all remaining terms strictly alternate;
4. All $P_n(u)$ with n even consist of even powers of u and end in either 1 or -1, alternatively;
5. All $P_n(u)$ with n odd consist of odd powers of u and end with a coefficient equal to either n or $-n$;
6. The sum of the coefficients in all cases is always equal to 1.

In order to investigate additional relationships that exist among the coefficients, it is convenient to display these numbers in triangular arrays analogous to Pascal's Triangle. We begin with the array of all coefficients, as shown in Figure 1. Examining the entries in Figure 1, we see that each interior entry is precisely equal to twice the entry diagonally above it to the right minus the entry two rows up and one position to the left. For instance, the 18 in the sixth row is obtained from $2(5)-(-8)$.

$$\begin{array}{ccccccccclr}
1 & 0 & & & & & & & & & n=1 \\
2 & 0 & -1 & & & & & & & & n=2 \\
4 & 0 & -3 & 0 & & & & & & & n=3 \\
8 & 0 & -8 & 0 & 1 & & & & & & n=4 \\
16 & 0 & -20 & 0 & 5 & 0 & & & & & n=5 \\
32 & 0 & -48 & 0 & 18 & 0 & -1 & & & & n=6 \\
64 & 0 & -112 & 0 & 56 & 0 & -7 & 0 & & & n=7 \\
128 & 0 & -256 & 0 & 160 & 0 & -32 & 0 & 1 & & n=8 \\
 & & & & \vdots & & & & & &
\end{array}$$

Figure 1. Triangular Array of Coefficients

In general, we obtain the recursion relation

$$A_{n,k} = 2A_{n-1,k} - A_{n-2,k-2} \tag{4}$$

for all $n \geq 2$ and all $k = 2, 3, ..., n$.

It is now possible to obtain a set of closed form expressions for the nonzero terms in each of the diagonals (lower left to upper right) in Figure 1 and, hence, for the coefficients in the polynomial $P_n(u)$. The terms in the first several upward diagonal positions satisfy:

$$\begin{array}{ll}
A_{n,2} = -\,\mathrm{n}\, 2^{n-3} & n \geq 2 \\
A_{n,4} = n(n-3)\, 2^{n-5}/2 & n \geq 4 \\
A_{n,6} = -\,n(n-4)\,(n-5)\, 2^{n-7}/3! & n \geq 6 \\
A_{n,8} = n(n-5)(n-6)(n-7)\, 2^{n-9}/4! & n \geq 8
\end{array}$$

and hold for all the indicated values of n as well. These formulas clearly indicate the forms for the corresponding formulas for $A_{n,k}$ for $k = 10, 12, 14, ...$ All of these restuls can be proven using induction. For instance, to prove the result for $A_{n,4}$, we apply the recursion relation (4) to find

$$\begin{aligned}
A_{n,4} &= 2A_{n-1,4} - A_{n-2,2} \\
&= 2[(n-1)(n-4)\, 2^{n-6}/2] + (n-2)\, 2^{n-5} \\
&= 2^{n-6}\,[(n-1)(n-4) + 2(n-2)] \\
&= 2^{n-5}\,[n^2 - 3n]/2 \\
&= n(n-3)\, 2^{n-5}/2\,.
\end{aligned}$$

We note that there are also some equally interesting closed form expressions for the terms in the downward diagonals (upper left to lower right). To find them, construct two other triangular arrays containing just the absolute values of the nonzero entries, one for the even values of n and another for the odd values. The results for n

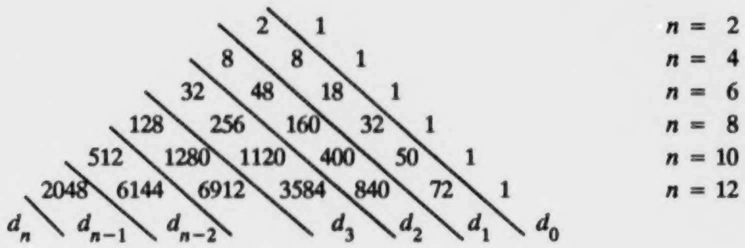

Figure 2. Triangular Array of Nonzero Coefficients for even *n*.

even are shown in Figure 2. For these downward diagonal terms, we have the following relationships:

$d_1 = n^2/2$ $\qquad$ n even, $n \geq 2$

$d_2 = n^2(n^2-4)/4!$ $\qquad$ n even, $n \geq 4$

$d_3 = n^2(n^2-4)(n^2-16)/6!$ $\qquad$ n even, $n \geq 6$.

The corresponding relationships for odd values of n are:

$d_1 = n(n^2-1)/3!$ $\qquad$ n odd, $n \geq 3$

$d_2 = n(n^2-1)(n^2-9)/5\,!$ $\qquad$ n odd, $n \geq 5$

$d_3 = n(n^2-1)(n^2-9)(n^2-25)/7!$ $\qquad$ n odd, $n \geq 7$

From these expressions, it is evident what the patterns are for the subsequent terms along the downward diagonals in both triangular arrays.

For those readers wishing to explore the further relationships, we suggest utilizing the trigonometric identity (2) to generate additional expressions. We note that this can be implemented particularly easily using a symbolic manipulation software package such as muMath (1986).

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Acknowledgements

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A Useful Vector Decomposition Theorem

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Tony Labarre has taught at CSU Fresno since 1961. He received his Bachelor of Engineering and MS degrees from Tulane University and his PhD from the University of Oklahoma. His current mathematical interest is in differential geometry.

Unifying themes should be used by mathematics instructors to help students see the relationships among topics in a particular course (or courses). For example, the idea of a limit serves as a unifying theme for the study of calculus. A theme that I have found to be particularly helpful in all of my classes that utilize the concept of a vector is the idea of the decomposition of vectors. I introduce this theme in the second semester of an introductory calculus course that is offered at CSU Fresno for students who lack a strong high school mathematics background. Traditional calculus students also could benefit from a knowledge of the material that follows.

Let $A = (a_1, a_2)$ be a plane vector. If we rotate A clockwise through 90° we obtain a new vector that I will call "A perp" (see Labarre, 1983), denoted by $A^\perp$, and given by

$$A^\perp = (a_2, -a_1).$$

Since the dot product

$$A \cdot A^\perp = (a_1, a_2) \cdot (a_2, -a_1) = a_1 a_2 - a_2 a_1 = 0,$$

it is clear that the two vectors are orthogonal.

The operator that sends A into $A^\perp$ has the following convenient properties whose proofs are left as exercises. The arguments are straight-forward and constitute good examples of proofs that can be given to elementary mathematics students.

1. $(A + B)^\perp = A^\perp + B^\perp$ (the operator is **additive**),
2. $(kA)^\perp = kA^\perp$ (the operator is **homogeneous**),
3. $(A^\perp)^\perp = -A$,
4. $|A^\perp| = |A|$,
5. $A \cdot B^\perp = -A^\perp \cdot B$.

Note: Properties (1) and (2) say "perp" is a **linear operator** and this accounts for the ease in manipulating the operator.

The Decomposition Theorem

Consider two plane vectors A and B. Let us decompose A into the sum of two vectors X and Y

$$A = X + Y, \tag{1}$$

where X is a vector along B, (i.e., X is a multiple of B) and Y is perpendicular to B, (i.e., Y is a multiple of $B^{\perp}$).

The problem can be formulated in an equivalent way (see Figure 1): Given A and B, find two real numbers r and s such that

$$A = rB + sB^{\perp}. \tag{2}$$

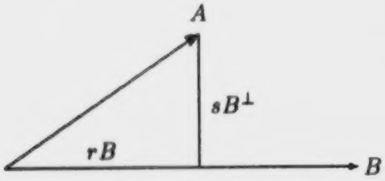

Figure 1. Decomposition of A in terms of B and $B^{\perp}$.

To find r, dot both sides of (2) by B:

$$A \cdot B = rB \cdot B + sB^{\perp} \cdot B, \qquad \text{or}$$

$$A \cdot B = rB \cdot B. \tag{3}$$

Assuming $B \neq O$, we have

$$r = \frac{A \cdot B}{B \cdot B}.$$

Returning to (2), we calculate s by dotting (2) by $B^{\perp}$ to obtain

$$s = \frac{A \cdot B^{\perp}}{B \cdot B}, \tag{4}$$

where we used the fact $B \cdot B = B^{\perp} \cdot B^{\perp}$. Therefore,

Theorem 1. (Decomposition of a vector). Let A be an arbitrary vector and let B be a nonzero vector in the plane. Then A can be decomposed into a sum of two vectors

$$A = \left(\frac{A \cdot B}{B \cdot B}\right) B + \left(\frac{A \cdot B^{\perp}}{B \cdot B}\right) B^{\perp}. \tag{5}$$

The decomposition given by (5) is unique. Indeed, uniqueness can be argued this way: If

$$A = rB + sB^{\perp}, \text{and if } A = pB + qB^{\perp},$$

then subtraction gives $O = (r-p)B + (s-q)B^{\perp}$. Dotting by B yields $r = p$, while dotting by $B^{\perp}$ yields $s = q$.

Note: The vector $\left(\frac{A\cdot B}{B\cdot B}\right)B$ in the decomposition (5) is called the **projection of A onto** ***B***, while the vector $\left(\frac{A\cdot B^{\perp}}{B\cdot B}\right)B^{\perp}$ is called the **projection of** ***A*** **onto** $B^{\perp}$.

Some Applications of the Decomposition Theorem

Trigonometric Functions. As a first application of the decomposition theorem, we introduce precise definitions of the trigonometric functions. To make the concept of angle precise, we think of the initial and terminal sides of the angle as being specified by nonzero vectors and we use an ordered pair of vectors (A,B) to distinguish the initial side (vector A) from the terminal side (vector B) (see Figure 2).

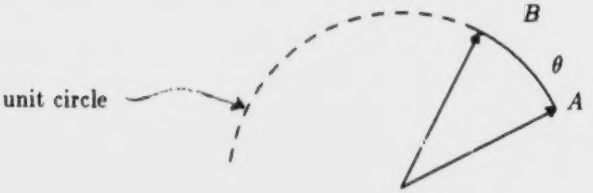

Figure 2. Angle (A, B) is an ordered pair of vectors.

Since the vectors we use are free vectors (i.e., the location of their initial points is immaterial), we can always represent them so they have the same initial point and this will serve as the vertex of the angle. Formally:

Definition 1. An **angle** is an ordered pair (A,B) of nonzero vectors.

A measure of an angle is found by rotating the initial vector about the vertex until it occupies the position of the terminal vector of the angle. Measures generated by counterclockwise rotations are taken as positive, whereas measures generated by clockwise rotations are to be taken as negative.

For given angle (A,B) the decomposition of the initial vector A into a vector along B plus a vector perpendicular to B as found in (5) is

$$A = \left(\frac{A\cdot B}{B\cdot B}\right) B + \left(\frac{A\cdot B^{\perp}}{B\cdot B}\right) B^{\perp} .$$

Multiplying both sides of this equation by $B\cdot B$, we have

$$(B\cdot B)A = (A\cdot B)B + (A\cdot B^{\perp})B^{\perp}, \tag{6}$$

and then dotting both sides of (6) by A we get

$$(B\cdot B)(A\cdot A) = (A\cdot B)(B\cdot A) + (A\cdot B^{\perp})(B^{\perp}\cdot A).$$

This simplifies to

$$|B|^2|A|^2 = (A\cdot B)^2 + (A\cdot B^{\perp})^2. \tag{7}$$

Using (7), we see that $A \cdot B$, $A \cdot B^{\perp}$, and $|A|\,|B|$ may be interpreted as the lengths of legs and hypotenuse, respectively, of a right triangle (see Figure 3).

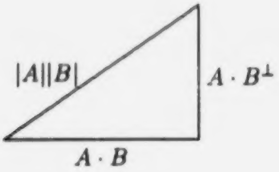

Figure 3. (Pythagorean Theorem) $|B|^2|A|^2 = (A \cdot B)^2 + (A \cdot B^{\perp})^2$.

Observe that although (7) was derived assuming $B \neq O$, nevertheless direct substitution of $B = O$ into (7) shows that it also holds in this case. Therefore,

Theorem 2. (Pythagorean Theorem). If A and B are arbitrary vectors, then $A \cdot B$ and $A \cdot B^{\perp}$ may be interpreted as the (signed) legs of a right triangle with hypotenuse $|A|\,|B|$, that is,

$$|B|^2|A|^2 = (A \cdot B)^2 + (A \cdot B^{\perp})^2.$$

Note that the vectors A and B enter equation (7) symmetrically; that is, if you interchange A and B, the equation remains valid. We call

$A \cdot B$ the **adjacent side** of angle (A,B),

$A \cdot B^{\perp}$ the **opposite side** of angle (A,B), and

$|A|\,|B|$ the **hypotenuse**.

The trigonometric functions of angle (A,B) are defined by

$$\cos(A,B) = \frac{A \cdot B}{|A|\,|B|} = \frac{\text{adjacent}}{\text{hypotenuse}};$$

$$\sin(A,B) = \frac{A \cdot B^{\perp}}{|A|\,|B|} = \frac{\text{opposite}}{\text{hypotenuse}};$$

$$\tan(A,B) = \frac{\sin(A,B)}{\cos(A,B)} = \frac{\text{opposite}}{\text{adjacent}}.$$

Perhaps one of the nicest aspects of these definitions is the fact that they are valid for an arbitrary angle (A,B) regardless of its location in the plane. The angle need not be in standard position. In spite of this simplistic approach, we have obtained the most general definitions! Indeed, the student need never modify the definition of the cosine (even for Euclidean n dimensional space). Moreover, our vector definitions contain the "standard angle position" and the "unit circle" definitions as special cases. Further advantages become evident as a trigonometry course is developed.

Example 1. The dot product $A \cdot B$ is the product of the length of A by the length of B by the cosine of the angle (A,B).

This fact is an immediate consequence of the definition of $\cos(A,B)$, for the definition implies

$$A \cdot B = |A|\,|B| \cos(A,B).$$

Example 2. Note that $\cos(A,B) = \dfrac{A \cdot B}{|A|\,|B|} = \dfrac{B \cdot A}{|B|\,|A|} = \cos(B,A)$; but

$$\sin(A,B) = \frac{A \cdot B^{\perp}}{|A|\,|B|} = -\frac{B \cdot A^{\perp}}{|B|\,|A|} = -\sin(B,A).$$

It follows that if θ is any measure of angle (A,B), then

$$\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta.$$

Example 3. The definition of $\sin(A,B)$ implies

$$A \cdot B^{\perp} = |A|\,|B| \sin(A,B).$$

On the other hand, we know that

$$\text{Area}\Delta OAB = \frac{1}{2}|A|\,|B| \sin(A,B).$$

Therefore

$$\text{Area}\Delta OAB = \frac{1}{2}A \cdot B^{\perp}.$$

The Cauchy-Schwarz Inequality. Focus your attention on (7). Take into account that $A \cdot B^{\perp}$ is a real number. Therefore, $(A \cdot B^{\perp})^2 \geq 0$. Hence

$$|A|^2|B|^2 \geq (A \cdot B)^2.$$

Taking the principal square roots, we have $|A|\,|B| \geq |A \cdot B|$. Therefore,

Theorem 3. (Cauchy-Schwarz Inequality). If A and B are any two plane vectors, then the product of their lengths is greater than or equal to the absolute value of their dot product:

$$|A|\,|B| \geq |A \cdot B|. \tag{8}$$

Sometimes it is useful to know when equality holds in (8). From (7) we see that this occurs if and only if $A \cdot B^{\perp} = 0$. From (6), we see that $A \cdot B^{\perp} = 0$ implies

$$(B \cdot B)A = (A \cdot B)B,$$

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which says that vector A is a multiple of B (remember, we assumed $B \neq O$). Even though this was obtained assuming $B \neq O$, inspection of this last equation shows that it holds for $B = O$ too. Therefore:

Corollary to Theorem 3. $|A \cdot B| = |A|\,|B|$ if and only if one of the vectors A, B is a multiple of the other.

Note: If $B \neq O$ and A happens to be the zero vector, then we cannot have B expressed as a multiple of A. However, in this case, $O = A = 0 \cdot B$, i.e., A is a multiple of B.

Signed distance from a line to a point. Suppose the given line has equation $ax + by + c = 0$. This can be written in vector form:

$$A \cdot X + c = 0, \text{ where } A = (a,b) \neq O \text{ and } X = (x,y).$$

Let P be the given point, and X be an arbitrary point on the line. Since $A \neq O$, we can decompose the vector $P - X$ into a vector along A plus a vector perpendicular to A (see Figure 4).

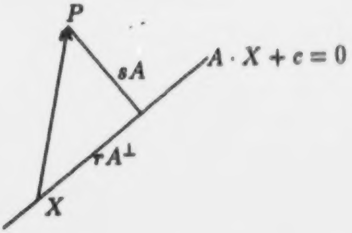

Figure 4. Finding the signed distance from the given line $A \cdot X + c = 0$ to the given point P

Since the vector A is perpendicular to the line, the decomposition of $P - X$ is

$$P - X = rA^{\perp} + sA, \tag{9}$$

and $|sA|$ is the required distance. Solving (9) for sA, we have

$$A \cdot P - A \cdot X = sA \cdot A, \tag{10}$$

$$s = \frac{A \cdot P - A \cdot X}{A \cdot A}. \tag{11}$$

Since $A \cdot X + c = 0$ implies $A \cdot X = -c$,

$$\begin{aligned} sA &= \frac{(A \cdot P + c)A}{|A|^2} \\ &= \left(\frac{A \cdot P + c}{|A|}\right) A^{*}, \end{aligned}$$

where $A^* = A/|A|$. If we keep in mind that A^* is a unit vector in the direction of A, this formula may be interpreted as follows: The numerical value of the number $(A \cdot P + c)/|A|$ gives the distance between the line $A \cdot X + c = 0$ and the point P; the perpendicular direction from the line to the point P is the same as the direction of A when the number $(A \cdot P + c)/|A|$ is positive and it is opposite to A when this number is negative.

Customarily, we "bury" the geometric information about the direction from the given line to the given point by insisting on a nonnegative number as our final answer. This is unfortunate because the computation of $\frac{A \cdot P + c}{|A|}$ takes less effort than the computation of $\frac{|A \cdot P + c|}{|A|}$ and it contains additional information!

Thus we say $(A \cdot P + c)/|\mathrm{A}|$ is the signed distance from line $A \cdot X + c = 0$ to the point P.

Cramer's Rule. As another application of the decomposition theorem, we will obtain Cramer's rule in a neat vector form. We consider the problem of solving simultaneously:

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 . \end{cases}$$

Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$. Then the system of equations becomes the single vector equation

$$xA + yB = C. \tag{12}$$

The problem is to find scalars x and y satisfying (12), where A, B, and C are given vectors. We know that this problem may be interpreted geometrically as the problem of finding the point of intersection of the two lines whose equations are given above. However, using a vector approach, (12) directs us to decompose vector C into a vector along A plus a vector along B. Thus C is to be the diagonal of a parallelogram whose sides are to be appropriate multiples xA and yB of A and B, respectively (see Figure 5).

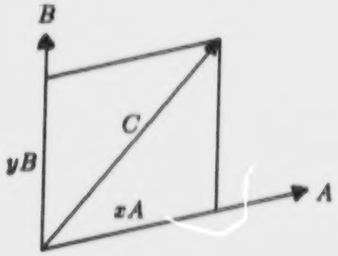

Figure 5. Cramer's Rule.

To find x, dot (12) with $B^{\perp}$:

$$xA \cdot B^{\perp} = C \cdot B^{\perp}, \text{ or}$$

$$x = \frac{C \cdot B^{\perp}}{A \cdot B^{\perp}}, \text{ assuming } A \cdot B^{\perp} \neq 0. \tag{13}$$

Remark 1. We leave it to the reader to check that (13) is simply a ratio of two determinants that are found in Cramer's rule for solving the simultaneous system.

Remark 2. According to the definitions of trigonometric functions given above, the condition $A \cdot B^{\perp} \neq 0$ means the area of triangle $OAB \neq 0$, i.e., A and B do not have the same orientation.

Next, to find y, dot product (12) with $A^{\perp}$ to obtain

$$y = \frac{C \cdot A^{\perp}}{B \cdot A^{\perp}}, \tag{14}$$

the other half of Cramer's rule. What we have shown is that if there are scalars x and y such that (12) holds, then they are given by (13) and (14). It is left to the reader to show that these values of x and y do indeed satisfy the vector equations (12). Summarizing the results, we have

Theorem 4. Cramer's Rule. If $A \cdot B^{\perp} \neq 0$ (i.e., if area $\Delta OAB \neq 0$), then $Ax + By = C$ has a unique solution given by

$$x = \frac{C \cdot B^{\perp}}{A \cdot B^{\perp}} \quad \text{and} \quad y = \frac{C \cdot A^{\perp}}{B \cdot A^{\perp}}.$$

Note that the vector decomposition theorem (Theorem 1) is a special case of the theorem just proved. To see this, keep in mind that the decomposition problem has the form: Given an arbitrary vector A and a vector $B \neq O$, find x and y such that $A = xB + yB^{\perp}$. To use Theorem 4 to solve the decomposition problem, make the following changes in the notation of the hypothesis of Theorem 4: Replace C by A, A by B, and B by $B^{\perp}$. We leave it to the reader to show that Theorem 4 then becomes the decomposition formula.

Reference

Labarre, A. E. (1983). *Elementary Real and Complex Analysis*. Fresno, CA: The Arnold Co.

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L'Hôpital's Rule with *Mathematica*

by John H. Mathews
California State University, Fullerton
Fullerton, CA 92634

John earned his PhD in mathematics from Michigan State University. He has been a member of the mathematics faculty of California State University for twenty-one years. He is the author of two books: Complex Variables and Numerical Methods.

Computer algebra systems (CAS's) are able to manipulate complicated algebraic quantities, perform symbolic differentiation and find limits. CAS's are a powerful assistant for keeping track of the messy details in multi-step problems which are prone to human error when done by pencil and paper. This note illustrates how the theoretical and computational aspects of L'Hôpital's rule can be enhanced with the use of a CAS. The computer algebra software *Mathematica* illustrated in this article is used in the computer laboratory of the School of Natural Science and Mathematics.

The Derivative Approach

Assume that F is differentiable on an open interval about the point $x = a$, then

$$F'(a) = \lim_{h \to 0} \frac{F(a+h) - F(a)}{h} \tag{1}$$

Similarly, the second derivative obeys:

$$\frac{F''(a)}{2!} = \lim_{h \to 0} \frac{F(a+h) - F(a) - F'(a)\,h}{h^2} \tag{2}$$

and, in general,

$$\frac{F^{(n)}(a)}{n!} = \lim_{h \to 0} \frac{F(a+h) - F(a) - F'(a)\,h - \ldots - \dfrac{F^{(n-1)}(a)}{(n-1)!}h^{n-1}}{h^n}. \tag{3}$$

Formula (3) can proven by using Taylor's formula with Lagrange's remainder and observing that the right side is $\frac{F^{(n)}(c)}{n!}$ where $c = c(h)$ lies between a and $a+h$. Hence, $h \to 0$ forces $c(h) \to a$ which establishes the result (see Schremmer, F. & Schremmer A., 1990).

In the case $F(a) = 0, F'(a) = 0, \ldots, F^{(n-1)}(a) = 0$ equation (3) becomes:

$$\frac{F^{(n)}(a)}{n!} = \lim_{h \to 0} \frac{F(a+h)}{h^n}. \tag{4}$$

We can start with formula (4) and prove L'Hôpital's rule. Suppose additionally that $G(a) = 0, G'(a) = 0, \ldots, G^{(n-1)}(a) = 0$ and $G^{(n)}(a) \neq 0$ and that $G(a+h) \neq 0$ and $G^{(n)}(a+h) \neq 0$ when $h \neq 0$ is small. Then applying (4) to F, G yields:

$$\lim_{h\to 0} \frac{F(a+h)}{G(a+h)} = \lim_{h\to 0} \frac{F(a+h)/h^n}{G(a+h)/h^n} = \frac{F^{(n)}(a)}{G^{(n)}(a)} . \tag{5}$$

L'Hôpital's rule enables us to compute the value of $\lim\limits_{h\to 0} \frac{F(a+h)}{G(a+h)}$ with the formula $\frac{F^{(n)}(a)}{G^{(n)}(a)}$ which "is determinable." For simplicity, the meaning of (5) is usually conveyed with a series of equations involving limits:

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \lim_{x\to a} \frac{F'(x)}{G'(x)} = \ldots = \lim_{x\to a} \frac{F^{(n)}(x)}{G^{(n)}(x)} . \tag{6}$$

The computer algebra software *Mathematica* can handle these constructs involving derivatives and limits. It is used by students in computer activities which help them "discover" and understand formula (3). The first derivative is investigated by issuing the *Mathematica* command for finding the limit of the difference quotient (1) is:

```
Limit[(F[a+h]-F[a])/h, h->0]
```

The **Limit** procedure in *Mathematica* assumes that higher derivatives of the arbitrary function **F[x]** exist, and the response by the computer is:

```
                                        F'[a]
```

The second derivative is involved in the limit of the difference quotient in (2):

```
Limit[(F[a+h]-F[a]-F'[a] h)/h^2, h->0]          F"[a]
                                                ------
                                                  2
```

When the quotients of two functions are involved, *Mathematica* will apply the above rules to find limits. For example,

```
Limit[(F[a+h]-F[a])/(G[a+h]-G[a]), h->0]        F'[a]
                                                ------
                                                G'[a]
```

If *Mathematica* is programmed with additional theoretical information, then it will imitate L'Hôpital's rule. The following dialogue will verify the $n = 1$ case. Clear F and G and teach *Mathematica* that $F(a) = 0$ and $G(a) = 0$ before taking the limit of the quotient $F(a+h)/G(a+h)$:

```
Clear[F,G]; F[a]=0; G[a]=0                      F'[a]
                                                ------
Limit[F[a+h]/G[a+h], h->0]                      G'[a]
```

Mathematica assumed that the values $F'(a)$ and $G'(a)$ were nonzero because it did not know otherwise. If we continue to instruct *Mathematica* about the nature of these functions then it will perform like a mathematician. For example, suppose that we define $F'(a) = 0$ and $G'(a) = 0$ before taking the limit, then we obtain:

```
F'[a] = 0; G'[a] = 0;                                   F"[a]
Limit[F[a + h]/G[a + h], h->0]                          ------
                                                        G"[a]
```

This is case $n=2$ of L'Hôpital's rule. These demonstrations illustrate that *Mathematica* has been programmed with the rudiments of analysis and uses the forementioned theory to compute limits.

Limits vs Function Evaluation

Let us illustrate *Mathematica's* powerful Limit procedure. For example, it can determine the value of

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{e^{2x}-\cos(2x)-2\sin(x)}{\sin(2x)\,(x+x^3)}. \tag{7}$$

First, enter the formulas for f and g into *Mathematica*:

```
f[x_] = Exp[2 x] - Cos[2 x] - 2 Sin[x]        E^2x - Cos[2 x] - 2 Sin[x]
g[x_] = Sin[2 x] (x + x^3)                    (x + x^3 ) Sin[2 x]
```

Then issue the command:

```
Limit[f[x]/g[x], x->0]                        2
```

Students soon discover that *Mathematica* can compute almost every limit found in elementary calculus. This leaves more time for students to develop an understanding of the theory surrounding the computation of limits. The first investigation is concerned with formula (6). As in the case of hand computations, the proper value of n must be determined experimentally, and is the smallest integer for which the computation $\lim_{x\to a}\frac{F^{(n)}(x)}{G^{(n)}(x)}$ exists. We can attempt to evaluate the quotient $\frac{f(x)}{g(x)}$ at $x=0$.

```
f[0]/g[0]                                     Indeterminate
```

Next, we can attempt to evaluate the quotient $\frac{f'(x)}{g'(x)}$ at $x=0$.

```
f'[0]/g'[0]                                   Indeterminate
```

Finally, we are able to successfully evaluate the quotient $\frac{f''(x)}{g''(x)}$ at $x=0$.

```
f"[0]/g"[0]                                   2
```

Therefore, we have found a "determinate value" for the limit in (6), i.e.

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{f''(0)}{g''(0)}=2.$$

The Graphical Approach.

For the example that was given in (7), we can easily obtain the graphs of the quotients $\frac{f(x)}{g(x)}$, $\frac{f'(x)}{g'(x)}$ and $\frac{f''(x)}{g''(x)}$ which are shown in Figures 1, 2 and 3, respectively. Although *Mathematica* draws all three graphs smoothly, they must be scrutinized carefully, because the first two pass through the point (0,2). This gives the appearance that the indeterminate quantities $\frac{f(0)}{g(0)}$ and $\frac{f'(0)}{g'(0)}$ take on the value 2. It occurs because *Mathematica* did not sample the value $x = 0$ when it drew the graph, and its smoothing algorithm filled in this point and gave the graphs the appearance of going through (0,2). The attempted calculations **f[0]/g[0]** and **f'[0]/g'[0]** are reported by *Mathematica* to be "Indeterminate."

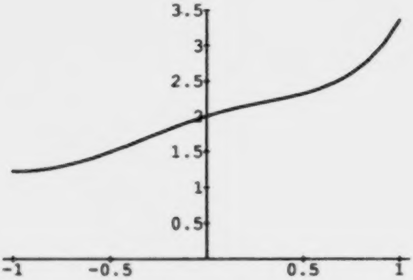

**Figure 1. Plot[f[x]/g[x],{x,-1,1},
PlotRange->{0,3.5}]**

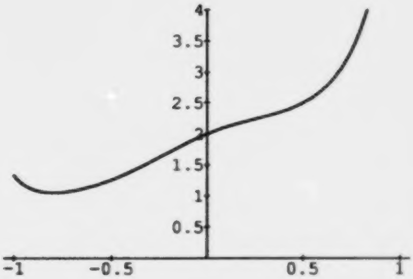

**Figure 2. Plot[f'[x]/g'[x],{x,-1,1},
PlotRange->{0,4}]**

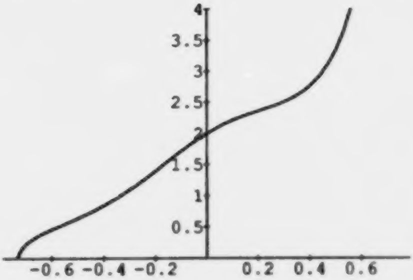

Figure 3. Plot[f" [x]/g" [x],{x,-0.75,0.75},PlotRange->{0,4}]

The Series Approach

The "lean and lively" calculus revitalization has produced several fresh approaches. The Lagrangian approach introduced by Francesca Schremmer and Alain Schremmer (1990) involves polynomial approximations for functions. The *Mathematica* command **Series** can be used to obtain polynomial expansions. We will illustrate the general case first, then carry out the details for the functions in the former example. The following commands create the Maclaurin series for the arbitrary function F(x) and G(x).

```
S[x_] = Series[F[x],{x,0,4}]
```

$$F[0] + F'[0]x + \frac{F''[0]x^2}{2} + \frac{F^{(3)}[0]x^3}{6} + \frac{F^4[0]x^4}{24} + 0[x]^5$$

```
T[x_] = Series[G[x],{x,0,4}]
```

$$G[0] + G'[0]x + \frac{G''[0]x^2}{2} + \frac{G^{(3)}[0]x^3}{6} + \frac{G^4[0]x^4}{24} + 0[x]^5$$

Since no other information is available, *Mathematica* will assume that F(0) and G(0) are nonzero. The limit of their difference quotient is:

```
Limit[S[x]/T[x], x->0]
```

$$\frac{F[0]}{G[0]}$$

Now if we require that F(0) = 0 and G(0) = 0, then the series will start with the terms F$'$(0) x and G$'$(0) x, respectively. Since the term x can be cancelled out of each term in the numerator and denominator of the quotient, the limit will behave as predicted by L'Hôpital's rule.

```
F[0]=0; G[0]=0;
```

```
S[x]
```

$$F'[0]x + \frac{F''[0]x^2}{2} + \frac{F^{(3)}[0]x^3}{6} + \frac{F^4[0]x^4}{24} + 0[x]^5$$

```
T[x]
```

$$G'[0]x + \frac{G''[0]x^2}{2} + \frac{G^{(3)}[0]x^3}{6} + \frac{G^4[0]x^4}{24} + 0[x]^5$$

```
Limit[S[x]/T[x], x->0]
```

$$\frac{F'[0]}{G'[0]}$$

The process can be extended to the case where L'Hôpital's rule must be applied twice. Since F$'$(0) = 0 and G$'$(0) = 0, the series will start with the terms $F''(0)x^2/2$ and $G''(0)x^2/2$, respectively. There will be a factor of x^2 in each term in the numerator and denominator which will cancel, and the limit is predicted by L'Hôpital's rule.

```
F'[0]=0; G'[0]=0;
```

```
S[x]
```

$$\frac{F''[0]x^2}{2} + \frac{F^{(3)}[0]x^3}{6} + \frac{F^4[0]x^4}{24} + 0[x]^5$$

```
T[x]
```

$$\frac{G''[0]x^2}{2} + \frac{G^{(3)}[0]x^3}{6} + \frac{G^4[0]x^4}{24} + 0[x]^5$$

```
Limit[S[x]/T[x], x->0]
```

$$\frac{F''[0]}{G''[0]}$$

Let us turn to our former example and investigate the behavior of quotient of the Maclaurin polynomials of degree six.

```
s[x_] = Normal[Series[f[x],{x,0,6}]];
t[x_] = Normal[Series[g[x],{x,0,6}]];
q[x_] = s[x]/t[x]
```

```
          5x^3    x^5    8x^6
   4x^2 + ---- + ---- + ----
           3      4      45
  ---------------------------
              2x^4   16x^6
       2x^2 + ---- - -----
               3      15
```

Observe that the quotient $q(x)$ is indeterminate at $x = 0$.

```
q[0]                                    Indeterminate
```

But the limit of $q(x)$ as $x \to 0$ exists.

```
Limit[q[x],x->0]                        2
```

The *Mathematica* command for factoring out x^2 is a little awkward. It is accomplished by typing:

```
r[x_] = PolynomialQuotient[s[x],x^2,x]/PolynomialQuotient[t[x],x^2,x]
```

```
        5x    x^3    8x^4
   4 + --- + ---- + ----
        3     4      45
  ------------------------
           2x^2   16x^4
       2 + ---- - -----
            3      15
```

The function $r(x)$ is continuous at $x=0$ and has a limit as $x \to 0$, as shown by the calculation:

```
r[0]                                    2
```

A comparison of the graphs $y = r(x)$ and $y = \frac{f(x)}{g(x)}$ in Figure 4 makes this point clear.

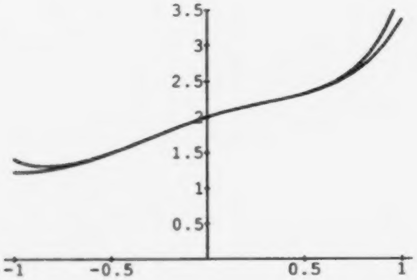

Figure 4. Plot[{f[x]/g[x],r[x]},{x,-1,1},PlotRange->{0,3.5}]

Other Indeterminate Forms.

The foregoing was aimed at showing that *Mathematica* reasons like a mathematician and can be of assistance with almost any process such as taking limits, evaluating derivatives, finding series or graphing. If finding the value of a limit is the goal, then *Mathematica* can serve as a practical tool. It is also capable of working with infinity. The following examples illustrate its power, and the reader can discover more.

Limit[(5x^3-2x^2+4x-5)/(2x^3+x^2-x+4),x->Infinity]	$\frac{5}{2}$
Limit[x/Log[x],x->Infinity]	Infinity
Limit[x^12/Exp[2x],x->Infinity]	0
Limit[x Log[x],x->0]	0
Limit[Log[1-2x]/Tan[Pi x],x->1/2]	0

Sometimes *Mathematica* cannot find a limit. If the quotient **x^12/Exp[2x]** in the third limit is changed to **x^13/Exp[2x]** then it will balk.

Limit[x^13/Exp[2x],x->Infinity] Limit::nlm: Could not find definite limit.

Limit [$\frac{x^{13}}{E^{2x}}$, x -> Infinity]

Every computer algebra has its limitations. However, this leaves many avenues to explore, debate and enjoy. The reader is encouraged to use a computer algebra system and investigate its potential for both research and teaching. Other popular systems are *Derive, Maple* and *Theorist*.

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SHORT COMMUNICATIONS

Compound Interest the e^x Way

by Bill Leonard and Harris S. Shultz
California State University, Fullerton
Fullerton, CA 92634

On a recent calculus test, the students were asked to find the amount of money in a savings account after five years if $1000 is deposited at 8% compounded monthly. The standard solution would be

$$A = 1000\left(1 + \frac{.08}{12}\right)^{(12)(5)} \approx \$1489.85.$$

One of the students offered the following solution:

1. Set $e^{r/12} = 1 + \frac{.08}{12}$
2. Solve for r: $r = 12\ln\left(1 + \frac{.08}{12}\right)$
3. Therefore, $A = 1000e^{5r} \approx \$1489.85$.

At first, the instructor thought the student was "just lucky." However, awareness of the student's strong ability suggested that a closer look was in order, and, indeed, his solution was correct. By setting $e^{r/12} = 1 + .08/12$, the student was asking what annual rate r compounded continuously yields the same amount of interest after one month as "simply" giving the investor 1/12 of 8% of his money? This is a less familiar version of effective annual yield. Specifically, the value of r computed is the annual rate for interest compounded continuously that is equivalent to 8% compounded monthly. While determining compound interest this way does not have any real computational advantage, the method does have some theoretical merit. The analysis of why the method works offers students a meaningful exercise in working with exponential and logarithmic functions. The validity of the method can be verified algebraically by noting that

$$e^{5r} = e^{5(12)\ln(1 + .08/12)} = e^{\ln(1 + .08/12)^{60}} = \left(1 + \frac{.08}{12}\right)^{60}$$

Name That Mathematician

by Charles Ashbacher
Hiawatha, IA 52233

The crossword puzzle on pages 50 and 51 is made up of the last names, single names in the case of ancients, of famous mathematicians throughout history. In some cases they are also known for the applications of their mathematics.

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> —Dale Varberg

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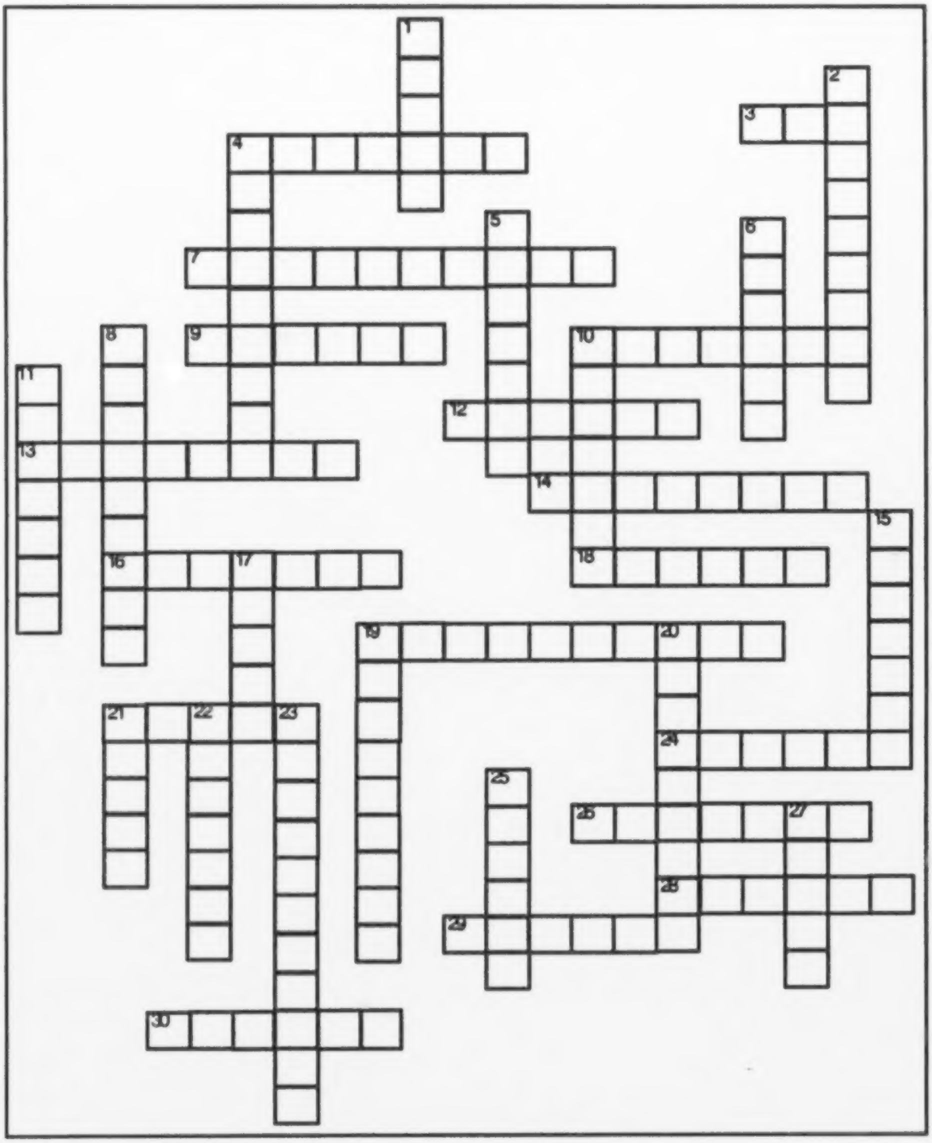
Name That Mathematician (continued from page 47)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ACROSS

3. The first computer programmer
4. All things are sine and cosine
7. His computer architecture is now losing favor
9. The first scientist of gambling
10. Killed by the acts of a saint
12. Found the "music of the spheres"
13. Synonymous with genius
14. His scissors are used to define the reals
16. The continental calculator as opposed to six down
18. His abstractions predate their mechanical counterparts.
19. Naked in the streets
21. His greatest success was in showing the inevitability of failure
24. All infinities are not alike
26. Mathematics as a recreation. Also, an eternal skeptic
28. People took his fifth apart
29. A swordsman whose field was fields
30. Very one-sided structure

DOWN

1. No half-empty or half-full to this bottle
2. His formulas are still being proven. Also, famous for a taxi
4. His rabbits can count many other things
5. Recently rehabilitated for what he saw in the sky
6. Stood on the shoulders of giants
8. First family of mathematics
10. Famous for a set of unsolved problems
11. The limit of his sum has few limits
15. She was interested in rings of a different sort
17. The laws that govern the thinker
19. Invented the computer, although few knew for a quarter of a century
20. x met y and they became analytic
21. His motto was "few but ripe"
22. Epitome of the Renaissance man
23. Moved the operator to front, from center
25. If only the margin was wider
27. A bridge became a theory

The solution to this puzzle is on page 64.

WANTED

"Good ideas can come in small packages."

I would appreciate your sending me your favorite brief mathematical quips and quotes so that I may use them for filler items for the journal.

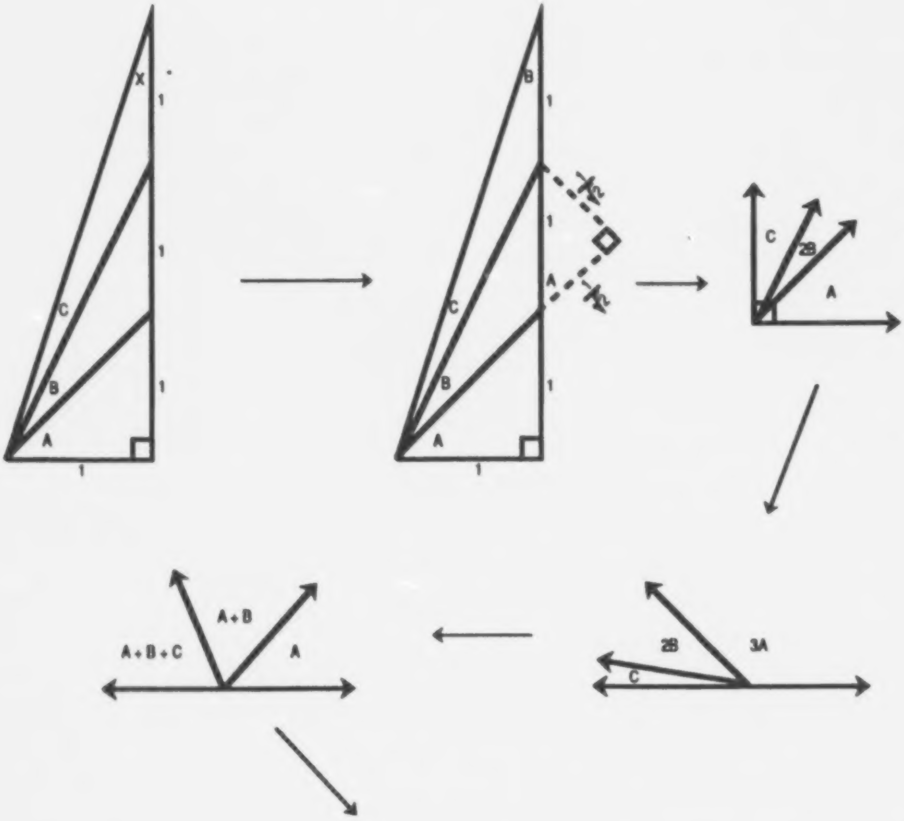
The editor

Mind's-Eye Proof

by Richard L. Francis
Southeast Missouri State University
Cape Girardeau, MO 63701

A mind's-eye proof is one in which geometric visualization, without appeal to words, provides the essential steps in a proof. The more profound the theorem proved as well as the more concise the visualized relationship, the more remarkable the proof. An example of a mind's-eye proof is that of

$$\mathrm{Tan}^{-1} 1 + \mathrm{Tan}^{-1} 2 + \mathrm{Tan}^{-1} 3 = \pi$$



$$\mathrm{Tan}^{-1} 1 + \mathrm{Tan}^{-1} 2 + \mathrm{Tan}^{-1} 3 = \pi$$

MATHEMATICS EDUCATION

A Program for the Math Anxious at the University of Minnesota

by Douglas F. Robertson
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Minneapolis, MN 55455

Douglas Robertson earned a BS in physics from Purdue University and an MA and PhD in mathematics education from the University of Minnesota. He has been teaching developmental mathematics and computing since 1974 through the General College, the open-admissions unit of the University of Minnesota.

The term *math anxiety,* popularized by Sheila Tobias and others during the mid-seventies, is used to describe the feelings of mental disorganization, fear, or even panic that prevent some people from effectively learning or working with mathematics (Tobias, 1980). Many people, including some mathematics teachers, have felt anxiety toward mathematics at one time or another, but each person's experience is uniquely his or her own. The causes of math anxiety are varied and usually occur before adulthood. Thus, by the time many students reach college age, they exhibit the symptoms of math anxiety and the problem interferes with their performances in situations where mathematics is required.

Some people are able to channel the feelings of anxiety into interest and excitement and the anxiety is not a serious problem. However, for the severely math anxious person, the excitement crosses over into panic and the anxiety becomes debilitating. Emotions take over control of the mind and body and productive work grinds to a halt.

The symptoms exhibited by the math anxious may lead to a vicious cycle of math avoidance characterized by the following:

Phase 1. Intense emotional negative reactions to situations where numbers are required. This may be caused by unpleasant experiences with mathematics in the past such as the following (the quotations given are from students):

- having poor mathematics teachers ("My fourth grade teacher could not do the new math problems either!");
- being embarrassed in or out of school in mathematics settings ("My teacher would make us work out problems at the board and throw the eraser at us when we made a mistake.");
- having unsympathetic or insensitive teachers, friends, or parents ("My father used to help me with my arithmetic—he would watch me do a problem and then whack me on the hand with a ruler when I made an error.");
- never being able to do mathematics problems correctly ("I never got above a 70% on any math test in my life.").

These feelings lead to . . .

Phase 2. Avoidance of situations in which mathematics is a part. People who dislike working with numbers will naturally try to avoid situations where numbers must be used. This avoidance leads to . . .

Phase 3. Poor preparation in mathematics. Obviously, if a person avoids mathematics, she or he will not get much practice or try to learn new mathematics. This lack of preparation leads to . . .

Phase 4. Poor performance in mathematics. As with any other physical or cognitive process, people who rarely work with mathematics will not be able to do it when they have to. This lack of ability to perform, of course, leads to . . .

Phase 1. Intense emotional negative reactions . . . and so on.

This cycle is often repeated over and over until the math anxious person completely adjusts his or her life so that mathematics is avoided no matter what the cost. By the time adulthood is reached, the math anxious person may be convinced (with good evidence) that he or she cannot do mathematics. Without intervention, the cycle is rarely broken.

The Math Anxiety Program of the University of Minnesota

The Math Anxiety Program of the University of Minnesota was designed to help math anxious adults learn basic mathematics. The program is offered in the evenings through the department of Continuing Education for Women (CEW). This department was created in 1960 for the purpose of relating higher education to the changing needs of women. Its original objective was to identify the educational needs of women and to provide the resources that would assist them in developing their abilities to the fullest. Today, CEW serves the educational needs of approximately 6,000 women and men each year. CEW is typical of the many specialized departments and colleges housed within the University whose purpose is to provide services to students with widely varying academic abilities and backgrounds. Indeed, through the General College, one of the fifteen colleges that comprise the University, arithmetic and elementary and intermediate algebra are taught in noncredit courses to approximately 2,700 students per year.

The CEW Math Anxiety Program had its beginnings in 1975 when an accounting professor and an economics professor came to CEW to discuss the academic problems being experienced by some of the women in their classes. These women were highly intelligent and motivated. Many had earned college degrees or had successfully completed some college work, but were having difficulty coping with the quantitative aspects of the accounting and economics courses. The students spoke of blocking out numbers, avoiding situations requiring the use of mathematics, not remembering how to do simple computations, and developing tension when faced with computations and testing.

The CEW staff investigated the situation by talking to the students and the course instructors, and by examining the available literature on the problems women have with mathematics. This research led to the development of the Math Anxiety Program in 1976 (Mathison & Robertson, 1980). Since then, over 5,000 students (80% of whom are women) ranging in age from 14 to 73 (average 28) have taken mathematics courses offered through the program. Approximately 82% of the current program participants are working full- or part-time, 14% are *regular* students (at the University or at other two-year or four-year colleges), and 14% classify themselves as homemakers. Currently the program offers courses in arithmetic, elementary and intermediate algebra, and introductory statistics. The CEW department supports the small administrative costs required to run the program and student tuition covers the cost of instruction.

Major Components of the Program

The CEW Math Anxiety Program consists of three parts: (1) a diagnostic clinic, where the anxiety and mathematics skill levels of participants are tested, the program is described, and the nature of mathematics is discussed, (2) a mathematics tutorial, and (3) a set of noncredit mathematics courses.

Program Component 1 – Diagnostic Clinic: These three hour clinics are offered three times per year to the general public as well as to specific groups or companies that request them, such as the State of Minnesota and Northwestern Bell Telephone. The purpose of the clinics is to provide participants with information concerning their mathematics skill levels and levels of math anxiety and to help them understand the nature of math anxiety and mathematics itself. In addition, the clinic serves to provide participants with a positive experience in a mathematical context by showing them that mathematics can be interesting, fun, and understandable.

The clinics generally enroll 20 to 30 participants and are lead by an experienced professor of developmental mathematics with extensive training in the mathematics education of adults.

At the beginning of the clinic, before any discussion takes place, participants are asked to write out their responses to the following four questions:

1. *What is your current level of skill in mathematics?*
2. *What is your definition of "math anxiety"?*
3. *What is your worst mathematics experience (if any)?*
4. *What is your best mathematics experience (if any)?*

Asking participants to write out their responses to these questions forces them to think about themselves in relation to mathematics. This serves as a good introduction for participants who knew that at least a part of their problem with mathematics had to do with their emotional feelings toward the subject. Participants are encouraged to share their answers to the questions with the group. As they hear each other's experiences, they begin to realize that they are not alone in their feelings of anxiety regarding mathematics. They begin to feel a little more comfortable about their problem and more enthusiastic about working towards a remedy. The clinic leader, who is also a teacher in the program, shares his own responses to the questions with the group.

After this discussion, the hour-long mathematics skills test is administered. Here it is important to convince participants that the test will not be used to evaluate them but to help them and the clinic leader, as partners, determine which mathematics course might be best for each individual. Participants are told that the test ranges in difficulty from arithmetic through intermediate algebra and that they will probably reach a point in the examination where they will not even understand the question being asked. At that point they should skip to the next question or stop. They are asked not to use calculators on the examination nor are they to make wild guesses. (Interested readers may obtain a copy of the examination from the author.)

As participants finish the test, they are given a sheet with the answers and a guide to the interpretation of their scores. After all participants have completed the test and graded their own work, a discussion of the interpretation of the scores is undertaken. It is important to reiterate that the examination is a placement instrument and not an intelligence test. An individual's score certainly does depend on her or his mathematics skill level, but it also depends on nonacademic factors such as fatigue, emotional feelings toward mathematics, and when she or he last worked on mathematics problems.

The placement test is used only as a rough guide and not as a perfect prescription. If participants score close to the cut-off for algebra, they have the choice as to which course to take. If they feel they know more than the test indicated, they can take algebra. If they feel that they were lucky on the test, or guessed at too many answers, they can take the arithmetic course. In any case, it is better to err on the side of taking a course which is too easy rather than one which is too difficult (i.e., it is better to be a little bored than to be completely lost and frustrated).

Sometimes a score falls *between the cracks* in the placement scheme. For example, some participants score well in algebra, but do poorly in arithmetic. It is necessary for a mathematics instructor to talk individually with those people to see what their specific backgrounds and future goals are. In some cases, it is not possible to determine with certainty which course should be taken and the instructor and participant have to make an educated guess.

It is important to have a mathematics teacher talk with participants about their choice of courses, either as individuals or as a group. The teacher is probably most qualified to help the participants determine which courses are most appropriate; it also helps the participants to interact with the teacher in a nonmathematical setting before the class begins. Many participants have stated that their former mathematics teachers are a part of, or the major cause of, their math anxiety. Interacting with a mathematics teacher in a counseling role helps dispel some of the fears that participants have concerning mathematics teachers.

Registering for a mathematics course is a big step for these people. If they can succeed in learning mathematics, many new educational and career opportunities may open for them. If they fail, their fear of mathematics will probably worsen and it may be years before they are willing to try again. For many, this is their last chance. Thus, the commitment to take a class must be carefully considered by both the student and his or her family. Students lacking the time, motivation, or support of their families have a low probability of being successful.

To help them make the right decision, the clinic leader and participants discuss what it is like taking a mathematics course at night and why anyone would want to do it. One cannot simply "add" a mathematics course to an already crowded life without something giving way. Most participants have jobs, family, extra curricular interests, other courses, etc., and simply do not have a lot of spare time. Spending one to two hours per day studying mathematics means that that time cannot be used for other activities. Students need to make a significant time commitment if they are to succeed in a mathematics class and this commitment must be shared by the members of their families. While studying mathematics at home, the students must be left alone for extended periods of time so that they can concentrate on their work. This is especially difficult for students who have young children or particularly demanding partners. Students lacking a strong commitment and the support of their families and friends are setting themselves up for failure.

After discussing the test results and describing the mathematics classes, a lecture/discussion is initiated on the uses of mathematical modeling as a problem solving tool and on mathematics as an interesting subject in its own right. The intent is to give participants the idea that mathematics is not just something that was invented to make their lives miserable, but that it was created by people to make life easier by helping them solve problems.

In the discussion, it is emphasized that there are many mathematical concepts which must be understood but that mathematics is also a language and has rules of grammar and a vocabulary which must be memorized. Every word and symbol has a meaning and must be taken into consideration when doing a problem. This kind of attention to detail is very difficult for many people, especially those who are *liberal arts* versus *engineering* types. To emphasize the care which must be taken when working with mathematics, participants are asked to respond to a number of *trick* questions such as "How many animals of each species did Moses take along on the ark?" (Copies of the clinic handouts are available from the author.) The mathematics lecture concludes with a brief discussion of different mathematics topics including modular art, Fibonacci numbers, and topology.

In the final part of the clinic participants are asked to complete the Math Anxiety Rating Scale (MARS – Richardson & Suinn, 1972). This is a ninety-question test which uses several subscales to measure the levels of anxiety participants feel toward mathematics. It is scored by the CEW office staff and the results are sent to participants a few days after the clinic.

Typically, two-thirds of the clinic participants register for a mathematics course after completing the clinic.

Program Component 2 – Mathematics Tutorial: In the mathematics tutorial, students are given the opportunity to work with a mathematics tutor on a one-on-one basis. It is crucial to hire tutors who have the ability to communicate with others and a sensitivity to the needs of others. When interviewing people who have applied for mathematics tutoring jobs, I ask them to help me with a problem I am having in adding fractions. That gives me a good idea of how they will react when students ask them similar questions. We have found that the best tutors are those who have completed the program themselves and then gone on to complete at least one course in calculus (unfortunately, there are not many of these students).

Program Component 3 – Mathematics Courses: It is difficult to imagine how anyone could overcome math anxiety without actually studying mathematics. People usually fear things they do not understand or things over which they have little control. If students can learn to use mathematics as a tool to help solve problems, they will be able to control it rather than allow it to control them. They may never like working with mathematics, but they will not have to fear it.

The Math Anxiety Program offers courses from arithmetic through intermediate algebra and one course in beginning statistical concepts. Generally, by the time students successfully complete intermediate algebra, they are in control of their debilitating anxieties toward mathematics and are in a position to make a choice, based on their own interests, about whether or not to continue on to college algebra and beyond. By this time the students have gained a respect both for mathematics and for their abilities to learn it and to use it as a tool in their daily and professional lives.

Description of the Arithmetic and Algebra Course

The majority of the students who participate in the program begin at the arithmetic or elementary algebra level. Even though the course we offer contains both components, students are encouraged to take either the arithmetic part or the algebra part, but not both at the same time. However, many students who are ready to study algebra also need to work on specific areas of weakness in arithmetic. By having the classes combined, students who need the algebra part of the course are able to attend those lectures on arithmetic which cover topics of interest to them.

The arithmetic and algebra portions of the class meet in adjacent rooms each Wednesday night for three hours. The arithmetic students are given ninety minutes of individualized attention from the tutor while the instructor discusses new material with the algebra students. The classes are then switched so that the algebra students can work individually with the tutor while the arithmetic students work on new material with the instructor. We have found that having the same instructor teach both the arithmetic and algebra courses helps students to appreciate the connection of the two subjects – indeed, the arithmetic course is taught as a special case of algebra using much of the same vocabulary, processes, and problem solving techniques. This makes the transition from arithmetic to algebra much smoother.

The lectures cover a wide range of topics, always within the context of new mathematical content, including the nature of mathematics, how to study mathematics, how to read mathematics textbooks, how to prepare for mathematics tests, what to do when feeling frustrated, where mathematics is used, how to use mathematics as a tool, tricks of the trade (both quantitative and social), and so on. The main idea woven throughout the course is that mathematics is a useful problem solving tool which can be learned by anyone who is willing and able to commit enough time and energy to the task.

Our students tell us that the single most important factor in the effectiveness of the course is the instructor. He or she must be sensitive to the mathematical and emotional needs of the students, make the course fun and nonthreatening, and give students the opportunity for success as soon as possible during the first class meeting.

Towards the end of the course, students are encouraged to sign up for further mathematics classes. Some want to "relax" for a quarter and not take another class right away. This is discouraged for two reasons: (1) they will forget a lot of the facts and procedures they have learned if they stop using them, and (2) the momentum they have generated in confronting their math anxiety will be lost. The fear and anxiety which the students have built up over many years will not be overcome in a single eleven-week course. Students should continue as long as their interest remains high and their needs have not been completely fulfilled.

Evaluation of the Program

The University of Minnesota completed three major evaluations of the Math Anxiety Program, one in 1977 (Hendel, 1977) just after the program began, another in 1981 (Lorenz & Hendel, 1981), and again in 1988 (Lindoo, 1988). The purpose of the evaluations was to determine the value of the program in helping students develop their mathematics skills and in helping them succeed in future mathematics courses, and to fine tune the program components to better serve the changing needs of the students. The results of the 1988 survey of 326 program participants (70% return rate) indicate that the program has been successful in developing students' skills in mathematics and in helping to prepare them for more advanced mathematics courses.

In the most recent program evaluation (Lindoo, 1988), the majority of the diagnostic clinic participants reported that the clinic was either very or somewhat useful in determining their levels of mathematics skills (90%) and their levels of math anxiety (84%), and in helping them decide which mathematics courses to take (85%).

Eighty-six percent of the students who completed the arithmetic and algebra course either took a higher level mathematics course or intended to do so in the future. Seventy-five percent of the students said that the arithmetic and algebra course helped them "a great deal" in their next mathematics class while 17% said it helped them "some".

Ninty-three percent of the students who took the arithmetic and algebra course reported that the course fulfilled their reason for taking it and 85% of the students reported that the course was helpful in dealing with their math anxiety. Reasons for taking the arithmetic and algebra course included preparation for credit-bearing mathematics courses (38%), preparation for present or future career opportunities (27%), or to help them in their personal lives (15%).

Students were also asked to comment on the program in general. A few typical responses are given below:

- "I am extremely grateful that the program is available. Until I heard that there was such as thing as math anxiety, I thought I was stupid. I was ashamed and avoided numbers at all cost."
- "I feel [the course] was *immensely* valuable for me."
- "If it had not been for the CEW Math Anxiety class and Professor Maren Watson, I would not have been able to move from social work to the business world—thanks a million."
- "Thanks for offering it! For the first time in my life I can work an algebra problem without shaking in fear."
- "The program gave me the confidence to deal with my anxiety. I only wish I would have been able to take it 20 years ago."
- "I found the program very beneficial and I now *enjoy* doing math—before I simply avoided it whenever possible."

Conclusion

Math anxiety is a serious problem for some adults. The Department of Continuing Education for Women at the University of Minnesota has developed a comprehensive program to help adult students who suffer from math anxiety to understand their problems and learn basic mathematics. The three components of the program – a diagnostic clinic, individualized tutoring, and content courses – are designed to analyze student needs, both mathematical and emotional, and to address those needs in a sensitive and humanistic manner. While developing such a program is time consuming, it is not difficult and, once the program is implemented, it is self-supporting.

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Additional Readings

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Solving Word Problems Using the Foreign Language Algebra

by Robert G. Pumford
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JCC
Jamestown

Bob earned a BA at Aquinas College in Grand Rapids, MI and an MA at Marquette University in Milwaukee, WI. He did additional study at the Center of Intercultural Formation, Cuernavaca, Mexico. He has taught at JCC for 23 years.

In January, 1963, I joined the faculty at Colegio San Carlos in Bogotá, Colombia, for a unique three-year teaching experience. This boys' high school, operated by the Benedictine Monks of North Dakota, used textbooks in English from the United States. Teaching mathematics in English was a challenge since although some students knew English very well, others who had just entered the school knew almost no English.

The students were at about the same academic level as high school students in the States. Therefore, you can imagine how I felt when I began the first set of word problems. I was not very optimistic. How could I expect these boys to solve word problems if students in the States, who knew English very well, struggled with them? What my San Carlos students did with these problems was amazing. They were exceptional! Each class in my three years in Bogotá handled word problems with ease. Why? I didn't know.

Naturally, I thought about San Carlos whenever I started word problems here in the States. At Jamestown Community College we teach courses in elementary and intermediate algebra which means I teach the same word problems I did in Bogotá. About seven or eight years after returning from Bogotá, I finally hit on the reason why my boys at San Carlos were so good at working word problems in English. Because they did not know English well, they continually translated word problems into their native tongue of Spanish. They saw the English sentences in pieces; the puzzle had been solved.

There appear to be two main ingredients in solving word problems written in any language: you have to see a sentence in pieces, and you must be able to translate those pieces into the beautiful, universal language of algebra. No wonder my San Carlos boys were so good. With all their experience in translating, piece by piece, from English into Spanish, it was a very small jump for them to translate word problems from English into the foreign language of algebra.

In the 14 years since I made that discovery, I have been experimenting with different techniques in teaching word problems. The first set of problems I do involve percents, with the unknowns called n or p. First, I have the students read a word problem such as: "Twenty-eight is thirty percent of which number?" and translate it into algebra. Then I ask them to translate it back into English. Next, I write an equation such as

$57 = 0.48n$ on the blackboard, and I have the students write out the translation in English.

Let's take a few minutes and look at this "different" idea of translating from the foreign language of algebra into English. When I was studying Latin in high school, I discovered that it was much easier to translate from Latin into English than vice versa. Notice what we ask our students to do. We give the students phrases and sentences in English and show them how to translate them into algebra. That's the tough translation! **What we don't do is give them an expression or an equation in algebra and have them translate it into English.** That's the easier translation and that discovery has helped make solving word problems easier for my students.

After having my algebra students translate a few more problems like $24 = p(60)$ into English, we are ready to translate from English into algebra. But at this point I use a second very important discovery: today's students are **better listeners than they are readers.** We are dealing with students who listen to rock tunes and actually understand the words being sung. This ability can be used by them to overcome their poor reading skills. Dictate a sentence like "Forty-five is what percent of eighty?" and have them translate it into an algebraic equation in their notebooks. Then have a student read the translation and write $45 = p(80)$ on the board. Don't bother to work out the equation. This would destroy the continuity of the translation exercise. Finally, have the students open their texts and read several word problems and translate them from English into algebra. The ice has been broken, the style has been set; and they are ready for more difficult word problems.

Next, translate expressions like $5x$, $x+3$, $x-4$, $3x+7$, and $4-5x$ into English. It takes time, but it is time well spent. Most of our students are very confused about how to solve word problems. Translate an expression like $x+3$ into English for the student in three ways: "an unknown number, increased by three"; "the sum of an unknown number and three"; and "three more than an unknown number." Proceed to do the same with expressions like $x-5$, $3-x$, and $2x-5$.

The students are now ready for phase two: making up a word problem. Suppose that we want the answer to the problem to be 7. Ask the students to write equations such as $5(7)-10 = 3(7) + 4$; $5x-10 = 3x+4$. Then have them translate the open equation into a word problem which is suitable for an algebra text: "Find an unknown number if ten less than five times. . . ." After two or three (or 10 ?) of these, the students are ready for the next phase: receiving **dictation** from an algebra text. "Find an unknown number if ten less than five times the unknown number equals three times the unknown number, increased by four." Again the students will read back their translations for you to write on the board, but not work out. After several of these, the students are ready to open their texts and work out the even-numbered problems. They are amazed at how easy it is to set up the algebraic equations, and experience a feeling of optimism. Having translated word problems into English, and having transcribed dictation from their teacher they have laid the foundation for **reading** and **translating into algebra.**

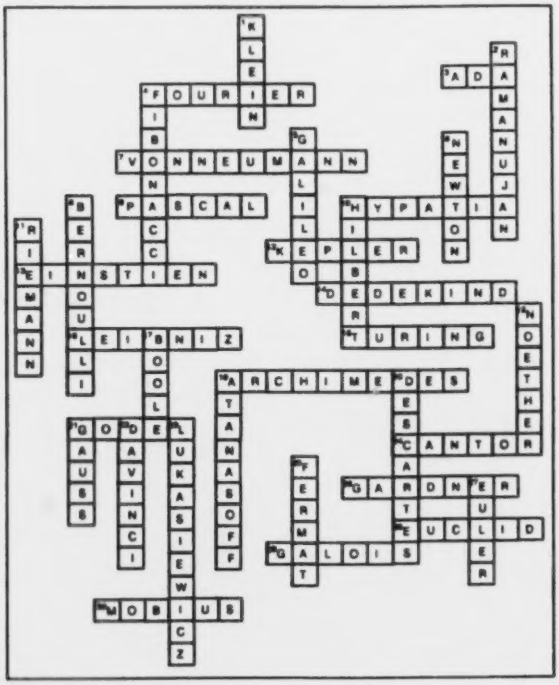
Conclusion

Our crowded curriculum tempts us to hurry and not take the time to lay a good foundation. How much time and energy do we then spend on remedial work? How many students become frustrated and give up? The approach outlined in this article has been very successful at all levels of algebra. I also taught word problems for the first time this past semester in a general mathematics class within the Developmental Studies Program. I discovered that translating from English to algebra and vice versa is an excellent way to assist poor writers to express themselves in an **organized manner**. It not only helps them to learn problem solving, it also helps the students to learn English grammar.

By translating from **algebra into English**, and receiving **dictation** in English and translating it into algebra the students are using powerful tools. Let's take advantage of the fact that most of our students have been raised as listeners. And please, do not mention anything to your students about starting a new, tough, better-pay-attention chapter on word problems; rather get them experiencing the new, exciting project of translating from algebra into English and vice versa before having them turn to the section on word problems in their text. Somehow students have developed a mental block toward word problems by second grade.

Invite your colleagues in English and foreign languages to do some translating from algebra and vice versa. Wouldn't it be exciting to walk past a classroom and hear an English teacher leading students in the translation of $3x-7=2x+6$ into a "word problem," or a Spanish teacher reading a "word problem" in Spanish and having the students translate it into algebra? This would be a natural way to integrate languages and algebra into the lives of our students. It would be a new twist to "writing across the curriculum."

Solution to **Name That Mathematician** crossword puzzle on page 50.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Educational Reflections

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This article[1] is an exploration of the theory of learning that has become known as constructivism. The beginnings of the theory currently labeled constructivism, its current status, and the possible changes in mathematics education that might occur if this perspective is taken are presented.

Beginnings

Tracking the beginnings of constructivist ideas is no easy task. The term has only recently become popular, but the ideas have been around for hundreds of years. Von Glasersfeld (1989b) points out that constructivist ideas started in pre-Socratic times when people questioned whether knowledge of the real world is certain or whether it is derived from experience. The idea of cognitive construction was first discussed in a Latin treatise on epistemology, by the Italian philosopher Vico in 1710 (Von Glasersfeld, 1988b).

The question of how knowledge is possible, or what is knowledge, moved through several philosophical views during the 17th and 18th centuries (Fabricius, 1983). The three major views were: (a) rationalism, (b) empiricism, and (c) romanticism. Rationalism was based on the idea that the entire world was rational and patterned after mathematical systems. Certain accepted premises enabled people to deduce conclusions that were considered facts about the world. The entire world was considered a mathematical model.

Empiricists did not use deductive reasoning in their view of the world, but rather inductive reasoning (Fabricius, 1983). They tried to generalize and predict based only on what was observed or experienced. Knowledge was based on perceptions of sight, sound, feel, and other senses.

Hume revealed the problems of empiricism and rationalism (Fabricius, 1983) in their inability to explain the structure of knowledge. If knowledge had to be deduced or induced, its structure could not be explained. This paradox is explained by Von Glasersfeld (1989b) by the fact that if true knowledge is representative of the world, then true knowledge cannot be tested. He writes:

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[1] Parts of this article appear in the author's doctoral dissertation, completed at The Ohio State University in 1991 under the direction of Alan Osborne, major advisor.

> If experience is the only contact a knower can have with the world, there is no way of comparing the products of experience with the reality from which whatever messages we receive are supposed to emanate.... The paradox then is this: to assess the truth of your knowledge you would have to know what you come to know before you know it. (cited in Narode, 1987, p. 22)

Rationalism and empiricism both place the structure of knowledge in objects. Rousseau (Fabricius, 1983) suggested that the structure of knowledge might not be far from the subject. He felt these structures could be found from within. Rousseau's view represented the opposite extreme. Knowledge was within the subject, not the object. This theory was romanticism.

In the late 18th century, Kant looked at a combination of subject and object (Fabricius, 1983), in particular, how objects appeared to the subject. He argued that the structure of knowledge was in neither subject nor object, but in the experience created by their interaction. His ideas conflicted with rationalism, empiricism, and romanticism. Kant was the first to conceive of this idea. It was a major breakthrough in what has become known as "constructivism."

Current Status

More recently, the work of Piaget has been studied with attention to the constructivist nature of his theories. Piaget studied learning theory from a developmental point of view and stated that experience is assimilated and accommodated in order to be known (Piaget, 1954). The subject, or learner, seeks a state of equilibrium and knowledge is created in the process. As Von Glasersfeld (1989a) explains:

> Knowledge for Piaget is never (and can never be) a representation of the real world. Instead it is a collection of conceptual structures that turn out to be adapted or viable within the knowing subject's range of experience. (p. 125)

This leads to current attempts to define or explain constructivism.

Von Glasersfeld (1989b) distinguishes between radical and trivial constructivism. The radical constructivist, as Von Glasersfeld calls himself, gives up all ideas that knowledge must somehow match the world. The radical constructivist claims: "perception and all forms of seeing . . . are the result of operations that have to be carried out by an active subject" (p. 22). The trivial constructivist talks about construction of knowledge, but still believes it represents reality. This causes confusion when reading about and discussing constructivism. The reader must be aware that both types of constructivist viewpoints exist and learn to distinguish one from the other.

Narode (1987) says constructivism is the process of individuals defining their own, constructed, worlds. Blais (1988) says "knowledge is something learners must construct for and by themselves. . . . Discovery, reinvention, or active reconstruction is necessary" (p. 627). He claims constructivists distinguish information, that can be told or given to someone, from knowledge, that requires gaining expertise. Piaget (1973) states a similar idea:

> To understand is to discover. . . . The goal of intellectual education is not to know how to repeat or retain ready-made truths. It is in learning to master the truth by oneself. (p. 106)

Kamii (1985) sees constructivism as the theory that a child builds his own knowledge and it comes from inside. Brooks (1987) thinks constructivism explains how viewers "come to know their own world" (p. 66).

Orton (1988) explains that constructivists consider the processes learners go through to create representations or relationships between ideas and that process involves reflection. Procedural knowledge is active and therefore preferred by constructivists. Von Glasersfeld (1988a) says the test of knowledge "is not whether it matches the world . . . but whether or not it fits the pursuit of our goals" (p. 2). Von Glasersfeld (1988a) also brings to our attention the following:

> Constructivism is a way of thinking which, at this stage in the development of our ideas, seems the most adequate; like all we call knowledge, it is not, and cannot be the description of an ontological reality and therefore it may change as our ways of experiencing and our purposes change. (p. 5)

So, constructivism has undergone and will undergo many changes.

Possible Effects on Mathematics Education

The National Research Council (1989) summarizes research on learning as follows:

> Research on learning shows that most students cannot learn mathematics effectively by only listening and imitating. . . . that students actually construct their own understanding based on new experiences that enlarge the intellectual framework in which ideas can be created. . . Much of the failure in school mathematics is due to a tradition of teaching that is inappropriate to the way most students learn. (p. 6)

The Commission on Standards for School Mathematics of the National Council of Teachers of Mathematics (1989) notes that:

> . . . learning does not occur by passive absorption. . . individuals approach a new task with prior knowledge, assimilate new information, and construct their own meanings. . . . This constructive, active view of the learning process must be reflected in the way much of mathematics is taught. (p. 10)

The Commission on Teaching Standards for School Mathematics (1991) describes the learning of students with the following:

> All students engage in a great deal of invention as they learn mathematics; they impose their own interpretation on what is presented to create a theory that makes sense to them. Students do not learn simply a subset of what they have been shown. Instead, they use new information to modify their prior beliefs. As a consequence, each student's knowledge of mathematics is uniquely personal. (p. 2)

All of the above incorporate a constructivist viewpoint on learning and teaching. This viewpoint requires a change in the way mathematics is taught that will, in turn, lead to changes in the curriculum.

The Mathematical Sciences Education Board (1990) states, as one redirection of the mathematics curriculum, the following, "The teaching of mathematics is shifting from an authoritarian model based on transmission of knowledge to a student-centered practice featuring stimulation of learning" (p. 5). This redirection will involve drastic changes in methods of teaching and the role of the teacher in the classroom.

Brooks (1987) says the role of the teacher in a constructivist classroom is one of mediator, not implementor. Teachers are developers and deliverers of curriculum. This changing, multi-dimensional role of the teacher is also presented by the Mathematical Sciences Education Board (1990) as follows:

- A role model who demonstrates not just the right way, but also the false starts and higher-order thinking skills that lead to the solution of problems;
- A consultant who helps individuals, small groups, or the whole class to decide if their work is keeping to the subject and making reasonable progress;
- A moderator who poses questions to consider, but leaves much of the decision making to the class;
- An interlocutor who supports students during class presentation, encouraging them to reflect on their activities and to explore mathematics on their own;
- A questioner who challenges students to make sure that what they are doing is reasonable and purposeful, and ensures that students can defend their conclusions. (p. 40)

Assuming these new roles in the classroom will not be an easy task, changes in methodology and teachers' roles must occur at all levels. The teaching of mathematics in the first two, as well as subsequent, years of college must change as well. Prospective teachers will long remember the ways they *were* taught rather than the lectures on the ways they *should* teach. As mathematics educators we must model these changes in methodology and the role of the teacher.

Conclusion

While many mathematics educators may agree with the basic tenants of constructivism, few will take action to change the teaching of mathematics. Agreement that our predominant methods of teaching mathematics are in conflict with the ways students learn is not sufficient. Attention must be directed to the research results on students' learning. Projects to test new methods of teaching and changes in the curriculum must be pursued. Mathematics educators must serve as role models in the classroom. Prospective and in-service teachers must be exposed to research results involving teaching and learning along with ways to incorporate changes into the classroom.

No purpose is served in agreeing that students construct or invent their own mathematics if we, as mathematics educators, do not capitalize on these theories and

change our approaches to teaching. We must encourage our students to appreciate and experiment with mathematics. We must recognize the varied ways they think and use mathematics. The constructivist viewpoint toward mathematics education will mean changes in the role of the teacher, a new respect for the learner, different approaches to mathematics education research, new teaching methods in the classroom, and changes in the curriculum.

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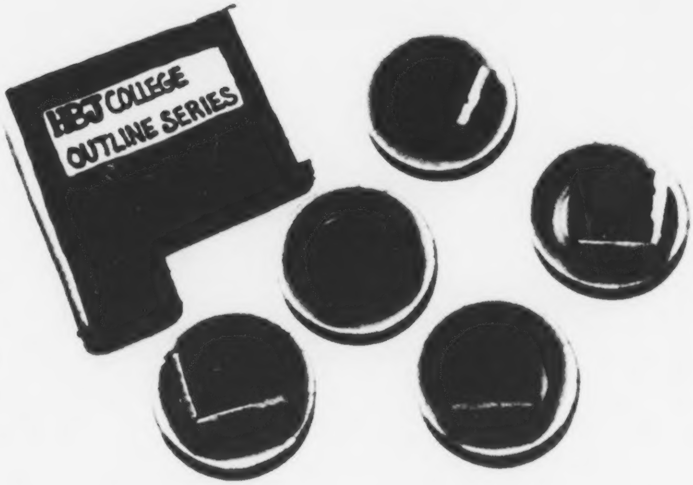

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Mathematics: An International View

Edited by Igor Malyshev and Joanne Rossi Becker

This column is as an avenue for discussion of the first two years of college mathematics with a global perspective. Because mathematics is an international culture, it is important to follow the development of mathematics education throughout the world. We feel that by sharing information from other countries with systems of education different from ours, we will gain an enriched understanding of, and a better perspective on, our own system. A completely different viewpoint on a topic can serve to stimulate our imaginations and help us find new solutions to a problem which will work in our context. As we move toward dramatic changes in both the curriculum and instruction in mathematics called for in many reports, reflection on practices used in other countries can only help our efforts.

For this column, we would like to publish two types of items: material with an international origin or context that you could use directly with your students; or information about the content and structure of coursework in other countries. We welcome submissions of 3-4 pages or suggestions of mathematicians from other countries from whom we could solicit relevant material.

Send contributions for this column to:

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Japanese Education Revisited

by Hidefumi Katsuura
San Jose State University, San Jose, CA 95192

Professor Katsuura received a BA from the College of Charleston and a PhD from the University of Delaware. He has taught at San Jose State since 1984. His research interest is in topology.

From July, 1990 to January, 1991, I was in Japan on a sabbatical leave from San Jose State University. While there as a guest scholar of Professor Sei-ichi Fukui at the Wakayama University, I had an opportunity to interview many professors and students. The Japanese educational system is not new to me. I was born and raised in a small agricultural community in Wakayama (near Osaka) until graduation from the local high school. Seventeen years ago, I entered a college in the United States. Since then I have lived in the United States. Returning to Japan proved to be a cultural shock. Many things had changed, yet others had not. The educational system is the one that changed very little.

The Ministry of Education controls and enforces the curriculum from the first to the twelfth grade. Students attend school for about 10 months of a year, spending approximately six hours in lectures Monday through Friday and four hours on Saturday. The Ministry controls the instruction in Japanese, English, mathematics, history, and science. Parents have no say in this matter. Hardly anyone repeats a grade. At the same time, no one is allowed to skip a grade. I found that most parents were content with this state of affairs. The main concern of the parents is probably to keep their children on the right track for college.

In an elementary school, first through sixth grade, the students spend about four to five hours a week studying mathematics and about one hour doing mathematics homework after each lesson. Besides mathematics, they have more homework in other

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subjects, especially Japanese. By the end of the sixth grade almost everyone can read and write and do arithmetic needed in daily life as an adult.

Teachers in elementary schools view learning as a group experience and they try to keep everyone in the group. But, what impressed me most was the level of parents' participation in the school activities. The main event of the fall was a sports event called the undokai, in which all the students participated, as did most of the parents and other members of the community.

In a junior high school, from the seventh through ninth grade, the curriculum becomes more rigorous. Students spend more hours, about five hours a week, studying mathematics. Before the end of the ninth grade, they start using linear equations and systems of equations to solve word problems. Additionally they learn to graph linear and quadratic polynomial functions, maximize and minimize quadratic functions by completing squares, prove theorems in Euclidean geometry using congruence and symmetry of triangles, and get an introduction to trigonometry. The gears are set for getting into the right high school, which will later lead to the right university.

At this stage, juku (after school privately-run tutorial classes) becomes a more important part of the learning experience. Many children have attended juku since preschool. At most jukus, students are taught on an individual basis. Some students spend time reviewing the materials with which they have difficulties, while others study regular subjects ahead of the school's curriculum. For better students, extra time is devoted to preparing for the high school entrance examination. Mathematics is the most popular subject at jukus. The ninth grade is an important transitional time in Japanese education because that is when compulsory education ends. To enter a high school, one must pass an examination which differs from school to school. The more prestigious (meaning a better chance of entering a major university) the school is, the harder the examination is. So the high school entrance examination can be viewed as the first test to determine which students will go to which college.

In Japan, there are essentially two types of high schools. The first is the normal high school (literal translation), which is for college-bound students. The other is the trade school, in which students learn practical skills in business, agriculture, and mechanics. About one-third of junior high school graduates attend trade school.

In a normal high school, mathematics is a required course for everyone, studied about seven hours a week. It is probably the most emphasized course in high school. Students have to learn a wide range of topics: trigonometric identities, solving inequalities, analytic geometry, differentiation and integration, mathematical induction, infinite series, complex variables, vector spaces, and probability theory. Most of these subjects are finished before the end of the eleventh grade. Studying these topics is one thing, learning to solve difficult problems in an examination is another matter. Students have to study an extensive number of problems, and they hope to recognize the problems or similar ones in entrance examinations. Almost the entire twelfth grade mathematics curriculum is devoted to preparing for these exams. (See the appendix for sample problems.)

The subjects covered in the university entrance examination depend on a student's major field of study. The examination usually consists of three to five subjects, and includes mathematics for science majors. For a state university, students usually have to take the mathematics entrance examination no matter what their major field is. The examinations for universities are different from school to school, and are offered at the location of the university once a year. Throughout Japan, **the entrance examination is usually the only factor considered by a university for admission.** Many Japanese view this as a fair system that gives everyone an equal opportunity to enter a prestigious university. However, I strongly disagree. It is very expensive to take an entrance examination, and attending a juku to help prepare for the exams costs at least $150 per month per subject.

Students have to choose their major field of study before entering the university. They are not allowed to change their major after admission, unless they quit the university and start all over by retaking the entrance examination. Transferring to another university is unthinkable. Even within a university, each school operates as a separate entity. It is not unusual to find, for example, a department of mathematics that serves only students in the school of science, another department of mathematics in the school of engineering, another for the school of education, and so on, in a single university.

Once these students are admitted to a university, the situation does not seem to improve. First, most of the students are already exhausted when they enter the university. College is considered as a place to have a four-year vacation after the period of examination turmoil and before the reality of working in Japanese industry. Even universities do not appear to expect much from students. It seems very easy to graduate from a university, prestigious or not. At Wakayama University, students are required to take about 120 units of courses and write a thesis in their major. However, units do not correspond to the number of hours that students attend classes. Typical freshlings and sophomores take 50 units a year. Each course counts as about three units. This means that each student takes about 16 classes per year. Classes meet approximately once a week for an hour and a half, and students complete most course work before they are juniors. For mathematics students, they learn introductions to analysis with $\varepsilon-\delta$ proofs, linear algebra, and group theory before their third year. Classrooms are frequently dull, with professors preaching, and students rarely asking questions. Classes are often canceled. Hardly any homework or quizzes are given, and only one examination per course per semester is required.

When students reach their junior year, they select thesis advisors in their major fields and specialize their study. Mathematics majors, for example, specialize in complex variables, algebra, analysis, or topology. Professor Fukui had three or four students studying complex analysis. When I observed his seminar, they were studying conformal mappings. They meet with the thesis advisor about once a week in a seminar. The seminars are the only courses in which they participate, and there are no more examinations. They seem to spend most of their time on part-time jobs to earn pocket money. In their senior year, students are concerned with the planning of a big "graduation trip" rather than working to graduate from the university or completing their thesis. Many seniors have contracts with future employers about one year before graduation.

Japanese industry does not expect much from college graduates either. By reading their job advertisements, one learns that most industries do not specify the specialists they need. Any major is accepted. To an industry, the name of a university is everything. A company usually retrains and re-educates the newly employed to suit its needs. Therefore, the entrance examination to a university is not only for the university. **It decides one's future employment.** A graduate from a famous university is set for life. In Japan, it usually means life-long employment with a promised promotion to a responsible position.

Appendix

The following are sample problems taken out of drill books for high school students:

1. If a and b satisfy the equation $ab=1$, prove that $a^2 + b^2 \geq a + b$.
2. If m, n are integers, prove that m^7n-mn^7 is a multiple of 42.
3. If the polynomial equation $x^3-px^2+(p-3)x+1=0$ has a root α express the other two roots in terms of α.
4. Let $S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$. Find: (i) $A = \lim_{n\to\infty} S_n$. (ii) $B = \lim_{n\to\infty} \frac{S_n}{\sqrt{n}}$.
5. Let $I(a,b) = \int_0^\pi (\sin x - a \cos x\, b \cos 2x)^2\, dx$. Find the values a, b that will minimize $I(a,b)$.

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Pandora's Rectangular Parallelepiped

Edited by $11 \times 28867 \int_{0}^{NER} SEC^{2}x \, dx$

First, an apology to Joe Browne. He has called my attention to the fact that an article written by him regarding acronyms (and, in particular, hypothetical acronyms involving AMATYC affiliates) had appeared in this *Review* prior to the PRP column in which I had discussed the same ideas. I have gone through my entire list of excuses, alibis and people-to-blame, and have yet to find a valid explanation of my oversight. I'll just have to say "Sorry, Joe!"

By the way, Joe was assigned to be my roommate at the AMATYC meeting in Dallas last October. When I mentioned that to the travel agent who had arranged for my hotel reservation, the response was "Joe Browne? Oh, sure! That's not very creative, but at least it's better than calling her John Smith."

Just the other day The Perfessor confronted me with a problem. He expressed his concern about a student in one of his classes who had recently made several references, both orally and in writing, to some sort of "community flaw." The Perfessor is both very civic minded and very shy, and so he asked me if I could discreetly probe into the matter. It didn't take long. In fact, the student himself seemed to be quite puzzled at The Perfessor's reaction to the student's finally having learned that A + B is always equal to B + A.

There is a question posed in the most recent issue of *Quantum*, a new science and mathematics journal for high school students, regarding the path of a racquetball hit at a certain direction and speed. That will undoubtedly cause millions of people to begin to ponder the general application of mathematics to that fine sport.

Actually, far more than mere vector calculus is involved in this game. Such things as randomness, fractals, number theory, and possibly even non-Euclidean geometry can be found there if one but applies oneself.

Let me explain. For example, court dimensions are strangely dynamic. Between games, the two side walls are found to be exactly 20 feet apart. Put the ball in play, however, and that distance ranges anywhere from 100 yards to 5 feet, depending upon whose turn it is – mine or my opponent's – to run after a ball at the other side of the court.

A fundamental feature of ordinary geometry is that the shortest distance between two points is a straight line, but the path that I frequently take between my current position and the position of the ball proves that to be quite incorrect. Usually, that path is best approximated by a non-simple, closed, polygonal curve; in heated rallies, it can even approach the boundary of the Mandelbrot set.

The speed of a served ball can mean the difference between winning and losing. World-class player Egan Inouye can serve at 192 miles per hour, but that is nothing – I can serve at 200 miles per hour; it just takes more serves (8 @ 25 mph) to do it.

Some unusual properties of the set of integers come into play when keeping score. (Incidentally, it is a fact that, in racquetball, points can be taken away from a player's score, even when the player has 0 points, resulting in a *negative* score; this is the only sport I know of in which that can happen.) The *rules* say that a single point is earned at a time. But many have been the times when, in one instant I am sitting on a comfortable 2-1 lead and in the very next instant my opponent announces the score to be 12-2 in his favor. This may have something to do with the space-time continuum.

Furthermore, there exists a peculiar relationship between the speed of a ball and the process of keeping score. This happens when ones opponent gets hit with the ball: at such times, the hittee has a tendency to forget not only his manners but also the score. A veteran player can add anywhere from 2 to 7 points to his own score, the actual amount added being directly proportional to the speed of the ball at impact.

Now aren't you glad the question was asked? Perhaps some of you readers have additional unusual applications of mathematics to racquetball or to other apparently non-mathematical human endeavors that you would like to share with colleagues. If so, you know where to send them!

Send contributions to this column to: Leslie R. Tanner, Editor, PRP Column, *The AMATYC Review*, c/o The College of Idaho, 2112 Cleveland Blvd, Caldwell, ID 83605

The Chalkboard

Edited by

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This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Send your contributions to either editor.

Several Rules Applied to One Problem

Beginning students in calculus are often confused by exposure to a variety of rules for differentiation in a relatively short period. The power, product, and quotient rules, and implicit differentiation are presented in quick succession and applied to numerous sample examples. I have found that students gain valuable insights into these methods when they are shown all the rules applied to the same problem.

As an example, let us consider finding the derivative for $y=x^6$. Of course, we can find the derivative by applying the power rule: $\frac{dy}{dx}=nx^{n-1}=6x^5$. But we can also apply other methods. For example, we can consider x as any of $x^5\cdot x^1, x^4\cdot x^2, x^3\cdot x^3$, and apply the product rule. Regarding y as $x^6\cdot x^0$ or $x^8\cdot x^{-2}$ before differentiating helps reinforce exponent concepts as well.

It is also instructive to obtain the same result using the quotient rule and treating y as $\frac{x^7}{x}$ or $\frac{x^8}{x^2}$. Obtaining the same result for the derivatives of $\frac{x^8}{x^2}$ and $x^8\cdot x^{-2}$ reinforces the concept that $x^{-2}=\frac{1}{x^2}$. Next the chain rule can be instructively applied. For example, $y=x^6$ may be considered as $(x^2)^3$, $(x^3)^2$, $(x^6)^1$, or $(x^1)^6$. Reinforcement of principles related to fractional and negative exponents can be obtained by differentiating $(x^{-3})^{-2}$ and $(x^{12})^{1/2}$. Finally, we can illustrate implicit differentiation concepts by considering $y-x^6 = 0$ or $y^3-x^{18}=0$.

When students see the same problem solved in so many ways, they are impressed. It increases their confidence in their ability to use the various rules of differentiation. A similar approach can be applied to many other problems. For example, $y=1$ may be treated as $x \cdot x^{-1}, x^3 \cdot x^{-3}, \frac{x}{x}, x^0$, etc., and $y = \frac{x^4-x^2}{x}$ may be considered as

$$(x^4-x^2) \cdot x^{-1} \text{ or as } x^3 - x .$$

Several benefits follow from this approach:

1. It shows students that mathematics is not just a collection of rules. Seeing that all the rules and methods of differentiation give the same result shows that mathematics makes logical sense.
2. It reinforces the notion that there is generally more than one way to solve a problem, and shows that if one method isn't successful, another should be tried.
3. It encourages students to seek the easiest way to solve a problem.
4. It may make students curious about whether there are several approaches to solving other problems (mathematical or other).
5. It helps reinforce student understanding of exponents.

Submitted by Richard H. Schwartz, College of Staten Island, Staten Island, NY 10301.

Going Over Homework

A difficult part of teaching mathematics is eliciting good questions from students about the assigned homework problems. Early in a mathematics course, I lay down some very firm ground rules about questions asked in class about homework.

One question that I will not accept during class is "Will you do problem 35?" (or any other specific problem). Although I am less rigid with the answer to this question in early algebra or other precalculus courses where the student may have some obvious anxiety about mathematics, the answer to this question in a higher level course is almost certainly "NO." I will, however, entertain such questions as "How would you start this problem?" or "I got to this point and did not know where to go next" or "I got the answer in the back of the book, but I don't know what it means."

It has been my experience that a motivated student is quite capable of completing portions of a problem even though the entire solution might not be within grasp. Besides, a wide assortment of complete problems are usually discussed within the classroom, and the only way a student learns how to solve problems is to solve more problems. Watching me solve a problem that has been solved several times before is not the answer. It must be made clear to the student, however, that such a response is not meant to avoid helping the student, but rather to channel the student's thinking to develop better questioning habits. There are more opportune times to work with the student individually, to review solved problems, or to help the student work through a problem.

Submitted by Ted Moore, Mohawk Valley Community College, Utica, NY 13501.

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Booth #14 AMATYC

The Problem Section

Edited by Michael W. Ecker, Ph.D.
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Assisted by Robert E. Stong, Ph.D.
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Charlottesville, VA 22903

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are those challenges and teasers that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematics.

Several of you have written in now to say that the problems have been getting a bit too hard. Skyline College's Ken Seydel reflected this well:

"Dear Mike: I was surprised to see the interior-point problem, since it has been around for a while, and has been in other problem columns. I agree with its inclusion, however, since I felt the problems have become much too esoteric lately. When are the people newer to the profession going to be introduced to the grand standard old problems?"

Thank you for the feedback, which is most helpful as well as appreciated!

There is, of course, a simple solution—to the problem of having difficult problems, that is—but it won't be easy: Everybody reading this should pledge to send at least one problem each time you get a new issue! This amounts to a mere two problems per year, which any math faculty member should be able to do standing on his or her head.

To submit material for this department, send your **new problem proposals** only, typed or printed neatly with separate items on separate pages, to the Problem Editor at the home address shown below. If you have a solution to your proposal, please include it as well, and/or any relevant comments, history, generalizations, special cases, observations, and/or improvements. Include a mailing label, self-addressed envelope, or self-addressed postcard if you'd like a reply.

All solutions to others' proposals should be sent directly to the Solutions Editor at his home address. Home addresses are:

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit, PA 18411

Dr. Robert E. Stong
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The AMATYC Review
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Please note this modification of policy: If you see your own problem printed later, please send Professor Stong a copy of your solution, regardless of whether you sent me one originally with your proposal or not. This is not mandatory as policy, but it is the only way to assure that your own solution will be considered for publication or given credit in the list of solvers.

Finally, a personal note of loss: On April 22, 1991, Samuel Yates – "Mr. Repunits" – a recreational mathematician and problem-solver, died. According to his brother, J.D. Yates, Sam was doing what he loved to do when a brain embolism struck; he was giving a lecture to a Mensa group on prime numbers, repunits, and Smith numbers.

I came to know Sam through correspondence over the past decade through several problem columns, including this one, and my own newsletter, *Recreational & Educa-*

tional Computing. His love for problems and mathematical recreations, especially pushing the frontier of number-crunching results about primes and the like, will live on, particularly in such publications as the *Journal of Recreational Mathematics.* Sam would surely be pleased to know his work will be remembered. But, I have been asked to relay one more message.

The family has set up a memorial fund for deserving graduate students in mathematics at Florida Atlantic University. Anyone wishing to contribute to this fund in his memory may send a check payable to the Samuel Yates Scholarship Fund. Send all contributions to either his widow (Mrs. Samuel Yates/157 Capri D, Kings Point/Delray Beach, FL 33484) or his brother (Mr. J.D. Yates/706 Foxpointe Circle/Delray Beach, FL 33445).

Teasers

In response to the need for diverse levels, starting this issue I inaugurate a new section: Teasers. These are problems that are a lot easier to handle, typically in a few minutes to at most an hour. Solutions will follow the next issue. Though this doesn't allow for the usual cycle of mail solutions to the Solutions Editor, I do solicit your own proposals as well as improved solutions to Teasers as published solutions appear. (Send to Problem Editor.)

Teaser #1: Proposed by the Problem Editor.

A natural number n is perfect if and only if the sum of its divisors that are less than n is n itself. Equivalently, the sum of all its divisors is $2n$. (There are about 30 known perfect numbers, the smallest is 6, the largest of which has 65,087 digits.) Prove that a natural number is perfect if and only if the sum of the reciprocals of all its divisors is 2.

Teaser #2: Proposed by the Problem Editor.

The batting average of a batter is the ratio of hits to times legally at bat. (This does not count walks or other such situations in which the outcome is neither a hit nor an out.) If a player's average is exactly .375, what is the smallest possible number of times the player could have been at bat?

New Problems

Set U Problems are due for ordinary consideration May 1, 1992. However, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Teasers – are always welcome.

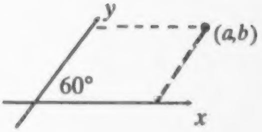
An asterisk on a problem indicates that the proposer did not supply a solution with the proposal.

Problem U-1. Proposed by Stephen Plett, Placentia, CA.

The repetitive nature of the polar graph $r = \cos(kt)$ is well-known for k an integer. Generalize by finding the smallest period of the graph for k rational.

Problem U-2. Proposed by Stephen Plett.

Suppose the usual two-dimensional x,y coordinate system is modified so that the positive y-axis is formed at a 60-degree angle to the positive x-axis. Consider any lattice point (a,b) in this system's first quadrant (i.e., a point (a,b) with integral coordinates and $a>0$, $b>0$). Each such point (a,b) determines a unique parallelogram formed by the two axes $(y=0, x=0)$, the line $x=a$, and the line $y=b$.



How many equilateral triangles are contained within or on this parallelogram if the triangles have lattice points for verticies and sides parallel to the axes and the lines $x+y$ = constant?

Problem U-3. Proposed by Ken Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Rationalize the denominator of F: $F = \dfrac{1}{1+\sqrt[3]{2}+\sqrt[3]{4}}$

Problem Editor's Comment: Thanks to my former student Keith Wandel for showing me this in October 1990, at which point I traced it back to Ken Boback, my colleague. Ken has a follow-up challenge: change "4" to "5" in the expression.

Problem U-4. Proposed by the Solutions Editor.

When the Cincinnati Reds won the 1990 World Series in four straight games, CBS may have lost as much as 150 million dollars in ad revenue. It was asserted that the average length of the series is six games.

a) Assuming the two teams are evenly matched in a best-of-seven series, what is the exact expected number of games played to obtain a winner?

b) Can the probabilities be chosen so that the expected length of the series is exactly six games?

Problem U-5. Proposed by Jayanthi Ganapathy, University of Wisconsin-Oshkosh.

Find all harmonic functions $u(x,y)$ (i.e., those satisfying $u_{xx} + u_{yy} = 0$) for which $u_{xx}u_{yy} - u_{xy}^2$ is identically zero. (This is not as hard as you think.)

Closing Comment: We now have a small storage of problems starting up. Please be patient if you recently submitted a problem and it has not appeared this issue.

Set S Solutions

Binomial Madness

S-1.* Proposed by Osvaldo Perez, Las Vegas, NV.

By exploration just a couple of years ago, I think I have come up with a new identity involving binomial coefficients, but I cannot prove it is true. Prove or disprove that, for real numbers x and y:

$$(x+y)^n = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \left(x-(k-1)y\right)^n .$$

Solutions by Joseph Chance, University of Texas - Pan American, Edinburg, TX; Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada; Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, WI; Christopher Stuart, New Mexico State University, Las Cruces, NM; Susan Timar, Centennial College, Toronto, Ontario, Canada; and Joseph Wiener, University of Texas - Pan American, Edinburg, TX

Letting $x + y = u$, $-y = v$, we rewrite the desired equation as

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (u+kv)^n = 0 .$$

Expanding $(u+kv)^n$ and taking the coefficient of $\binom{n}{j} u^{n-j} v^j$, one sees that this is equivalent to

$$\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} k^j = 0 \qquad \text{for } j = 0, 1, \dots, n.$$

Now, we have that

$$\sum_{k=0}^{n+1} \binom{n+1}{k} k^j x^k = D^j (1+x)^{n+1}$$

where $D = x(d/dx)$, and taking derivatives at most n times, $D^j(1+x)^{n+1}$ has a factor $(1+x)$. Substituting $x = -1$ will then give zero if $j \le n$.

Note: Bob Prielipp points out the reference F. Chorlton, "Dicing with finite differences," *Math. Gazette*, Dec. 1978, pp. 300-301 which gives the desired identity in only slightly different form.

Getting Seven The Hard Way

S-2. Proposed by the Solution Editor and one of his students.

A single die is rolled and a running sum is kept of the numbers that turn up. How many times do you theoretically expect to have to roll a single die to obtain a sum that is at least 7? Can you generalize to a single n-sided die with a sum of at least $n+1$? What happens as n increases without bound?

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada and the proposer. Partial results were obtained by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA and Frank Soler, De Anza College, Cupertino, CA.

Let $A(k,j)$ be the number of ways to obtain the sum j on k rolls of a die. Then

$$\sum A(k,j)x^k y^j = \sum_k x^k(y+y^2+\ldots+y^6)^k = x(y+\ldots+y^6)/[1-x(y+\ldots+y^6)].$$

Ignoring terms with degree more than 6 in y, we see that this is the same as

$$\frac{xy/(1-y)}{1-xy/(1-y)} = \frac{xy}{1-y(1+x)} = \sum y^j x(1+x)^{j-1}$$

and so $A(k,j)$ is $\binom{j-1}{k-1}$ if $j \le 6$. Therefore, the sum

$$\sum_{j\le 6} A(k,j) = \sum_{j\le 6}\binom{j-1}{k-1} = \binom{6}{k}$$

is the number of ways to have sum less than 7 on k rolls. The number of ways to get a sum which exceeds 6 on precisely the $(k+1)$st roll is then $6\binom{6}{k} - \binom{6}{k+1}$ and the expected number of rolls to get sum more than 6 is then

$$\sum \left(6\binom{6}{k} - \binom{6}{k+1}\right)(k+1)(1/6)^{k+1}$$

$$= \sum \binom{6}{k} k/6^k + \sum \binom{6}{k}(1/6)^k - \sum \binom{6}{k+1}(k+1)/6^{k+1}$$

$$= \sum \binom{6}{k} 1/6^k = (1+1/6)^6 .$$

If one rolls an n-sided die, the expected number of rolls needed to get the sum of the rolls to be $(n+1)$ or more is $(1+1/n)^n$, and as n gets large, this number approaches $e = 2.7128\ldots.$ To see this one need only replace 6 by n in the above.

A Tan Identity

S-3. Proposed by the Problem Editor.

Find all values of n for which the following holds true:

$$\mathrm{Tan}^{-1}(n) + \mathrm{Tan}^{-1}(n+1) = \mathrm{Tan}^{-1}(1/n) + \mathrm{Tan}^{-1}(1/(n+1)) + \pi/2 .$$

A remarkable pair of identities is concealed in the solution.

Solution by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Nicholas Belloit, Florida Community College at Jacksonville, Jacksonville, FL; Kenneth Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA; Joseph Chance, University of Texas - Pan American, Edinburg, TX; Margaret Cibes, University of Hartford, West Hartford, CT; Donald Fuller, Gainesville College, Gainesville, GA; Jayanthi Ganapathy, University of Wisconsin - Oshkosh, Oshkosh, WI; James Magliano, Union County College, Cranford, NJ; Stephen Plett, Placentia, CA; Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, WI; Susan Timar, Centennial College, Toronto, Ontario, Canada; Bella Wiener, University of Texas - Pan American, Edinburg, TX; and the proposer.

Looking at the right triangle with legs 1 and k, one sees that

$$\mathrm{Tan}^{-1}(k) + \mathrm{Tan}^{-1}(1/k) = \pi/2\,,$$

and the equation becomes (after using $k = n$ and $k = n + 1$, transposing, and dividing by 2):

$$\mathrm{Tan}^{-1}(n) + \mathrm{Tan}^{-1}(n+1) = 3\pi/4\,.$$

Taking the tangent of both sides of the equation gives

$$\frac{n+(n+1)}{1-n(n+1)} = -1 \quad \text{or} \quad n^2-n-2 = 0$$

Thus $n = -1$ (which is extraneous) or $n = 2$, which is the solution.

Note. The identities hinted at are the rule for $\mathrm{Tan}^{-1}(1/n)$ and the formulas

$$\mathrm{Tan}^{-1}(1)+\mathrm{Tan}^{-1}(2)+\mathrm{Tan}^{-1}(3) = \pi; \quad \mathrm{Tan}^{-1}(1)+\mathrm{Tan}^{-1}\left(\frac{1}{2}\right)+\mathrm{Tan}^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}\,.$$

Recurrence Occurrence

S-4. Proposed by Stanley Rabinowitz, Westford, MA.

Find a linear recurrence for the sequence whose n^{th} term is given by:

a) $u_n = x^n + y^n$,

b) $u_n = x^n + y^n + z^n$.

Solution by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Joseph Chance, University of Texas - Pan American, Edinburg, TX; James Magliano, Union County College, Cranford, NJ; Stephen Plett, Placentia, CA; and the proposer.

Solution I

Part (a): Multiply u_n by $x + y$ to get:

$$\begin{aligned} u_n(x+y) &= x^{n+1} + y^{n+1} + xy^n + yx^n \\ &= x^{n+1} + y^{n+1} + xy(x^{n-1} + y^{n-1}) \\ &= u_{n+1} + xyu_{n-1}\,. \end{aligned}$$

Thus, $u_{n+1} = (x+y)u_n - (xy)u_{n-1}$ is the desired linear recurrence.

Part (b): Multiply u_n by $x + y + z$ to get:

$$\begin{aligned} u_n(x+y+z) &= x^{n+1} + y^{n+1} + z^{n+1} + xy^n + xz^n + yx^n + yz^n + zx^n + zy^n \\ &= u_{n+1} + xyy^{n-1} + xzz^{n-1} + yxx^{n-1} + yzz^{n-1} + zxx^{n-1} + zyy^{n-1} \\ &= u_{n+1} + xy(x^{n-1} + y^{n-1}) + yz(y^{n-1} + z^{n-1}) + zx(z^{n-1} + x^{n-1}) \\ &= u_{n+1} + xy(u_{n-1} - z^{n-1}) + yz(u_{n-1} - x^{n-1}) + zx(u_{n-1} - y^{n-1}) \\ &= u_{n+1} + (xy + yz + zx)u_{n-1} - xyz^{n-1} - yzx^{n-1} - zxy^{n-1} \\ &= u_{n+1} + (xy + yz + zx)u_{n-1} - xyz(x^{n-2} + y^{n-2} + z^{n-2}) \\ &= u_{n+1} + (xy + yz + zx)u_{n-1} - xyzu_{n-2}. \end{aligned}$$

Thus, $u_{n+1} = (x + y + z)u_n - (xy + yz + zx)u_{n-1} + (xyz)u_{n-2}$ is the desired linear recurrence.

Solution II: With $s_n = \sum_{i=1}^{r} x_i^n$, a formula due to Newton says

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} + \dots + (-1)^t \sigma_t s_{n-t} + \dots + (-1)^n n\sigma_n = 0$$

where σ_i is the i-th elementary symmetric function of $x_1, \dots, x_r$. For the cases $r = 2$ and 3, this gives the recurrences

a) $u_n - (x + y)u_{n-1} + xyu_{n-2} = 0$ and

b) $u_n - (x + y + z)u_{n-1} + (xy + xz + yz)u_{n-2} - xyzu_{n-3} = 0$,

which can readily be verified.

Colorful Spaces

S-5. Proposed by Stephen Plett, Placentia, CA.

For $k > 1$, consider Euclidean space R^k with the usual distance. Can every point be assigned one of $k+1$ colors in such a way that no two points k apart will have the same color? You might wish to handle $k = 2$ and $k = 3$ first.

No solutions were submitted by readers. Here is the treatment by the Solutions Editor:

Consider a $(k+2)$-tuple of points $x_0, x_1, \dots, x_k, x_0'$, where the points $x_0, x_1, \dots, x_k$ and $x_0', x_1, \dots, x_k$ are $(k+1)$-tuples having the distance between any two points equal to k. Specifically, take the $(k+1)$-tuples to be the vertices of equilateral simplices with edges k, with the points x_0 and x_0' lying on opposite sides of the hyperplane spanned by the other vertices. If space can be colored with $k+1$ colors in the fashion desired, then the points of each $(k+1)$-tuples all have distinct colors, and hence x_0 and x_0' have the same color. Hence, every pair of points of space whose distance is that from x_0 to x_0' will have the same color. Since any two points in space can be joined by a sequence of points with any fixed distance between adjacent terms, all points of space must have the same color. This contradicts the coloring by $k+1$ colors.

Note. The case $k = 2$ was given as a problem on the Putnam exam and the solution may be found in the October 1989 issue of the *American Mathematical Monthly.*

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Book Reviews

Edited by John J. Edgell, Jr.

FOUNDATIONS OF DISCRETE MATHEMATICS by Albert D. Polimeni and H. Joseph Straight, Wadsworth Inc., Belmont, CA, 1990, 477 pages. ISBN 0-534-12402-X.

In the past several years, the different areas of mathematics that are combined into the discipline known as Discrete Mathematics have been fairly well standardized. However, the number of topics still remains large and it is not possible to include all of the topics in a standard one-semester course. Furthermore, there is always the difficulty regarding where to begin. Is this material to be taken before or after calculus and do you start with sets or some other foundation topic?

The book by Polimeni and Straight handles all of these questions well. Although the prerequisites are given as one year of college mathematics including a semester of calculus, careful choice of material would allow it to be used for a freshman level course. The writing is clear and the examples have sufficient explanation to allow the necessary amount of self-study. No programming is required and the computer examples are in positions that allow them to be skipped without loss of continuity.

The authors' choice of logic as the introductory topic works well and although there is more material than can be covered in a typical course, it is sufficiently independent to allow for selection. The chapter exercises are in a reasonable mix from trivial to difficult and many of the answers are found in an appendix. An instructor's manual is available with the remaining solutions.

No one teaching Discrete Mathematics should make a text selection without first scanning this book.

Reviewed by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA 52406

MATHEMATICS AND GENDER, (S) Elizabeth Fennema and Gilah C. Leder, Editors. Teachers College Press, Columbia University, New York, NY 10027. 1990, vii + 214 pp., paper ISNB 0-8077-3001-7, cloth ISBN 0-877-3002-5.

Equity is the most important reason for considering the relationship between gender and the study/teaching of mathematics. As Elizabeth Fennema points out in the introductory chapter, equity (and justice) involves not only equal educational opportunity, but also equal treatment, and equal outcome.

The editors, and four other contributors (Lindsay A. Tatre, Margaret R. Meyer, Mary Schatz Koehler, and Peter Kloosterman), provide a comprehensive view of the research into gender differences in mathematics over the past two decades. The cumulative bibliographies that they supply cover virtually every issue related to equity including general studies of motivation, attitude, and classroom activities, plus ample examples of research specifics. As might be expected, they seek explanations and even possible solutions to the problems of gender differences. To their credit, they do not offer a magic pill or simplistic answers. Nevertheless, by identifying differences that lead to success for more males than females, they shed light on attributes of good teaching and learning for all.

Reviewed by Leonard Feldman, San Jose State University, San Jose, CA 95192

Send Reviews to: Dr. John J. Edgell, Jr., Editor, Book Reviews, *The AMATYC Review*,
Mathematics Dept., Southwest Texas State University, San Marcos, TX 78666

Software Reviews

Edited by Shao Mah

Title: LINEAR-KIT
Author: Howard Anton
Distributor: John Wiley & Sons, Inc., 1 Wiley Drive, Somerset, NJ 08873
Systems Requirements: Apple IIe or IIc with 128k, single disk drive; also available for IBM PC.
Price: Apple or IBM version: $29.95

Linear-Kit comes with one 5-1/4 floppy disk and a user's manual. It is a software designed primarily as a learning aid or problem solving tool for students in a linear algebra course.

Linear-Kit can perform matrix operations in addition and subtraction of matrices, multiplication of matrices by a scalar, matrix multiplications, finding inverse and transpose of a matrix, reducing a matrix to its echelon form or reduced echelon form, calculating dominant eigenvalues and eigenvectors, and solving linear systems of equations. The Linear-Kit can also perform vector operations in addition, subtraction, calculating dot product and cross products, forming linear combinations of vectors, computing Euclidean norms, finding a basis for a spanning set of vectors, finding orthonormal basis for the space spanned by a set of vectors. The size of a matrix or a vector which can be used is relatively small. The vector can have at the most six components, and the maximum size of a matrix is six by six. However, the matrix or vector size should be adequate for students for doing their assignments or simple problem solving.

The best feature of Linear-Kit for students is that it can perform either rational model or decimal model of computation. By using rational model computation, a student can duplicate the examples in the textbook and follow the procedures of computations step by step. It is considerably helpful for students in self-learning of any vector and matrix problems. Linear-Kit can also be used for classroom demonstration so that one can show the problem solving techniques by projecting it onto the screen by computer video projection. In this way, the instructor can explain the vector or matrix operations to students step by step without doing tedious calculations by hand.

Linear-Kit is self-explanatory and very easy to use. The user's manual is also similar to a tutorial book. It contains all the information which the user might need to know and has examples in vector and matrix operations for every type of computation.

The only drawback of Linear-Kit is in its performing of different steps or types of operation in vector and matrix calculation. It uses the spacebar to advance forward to locate the proper operation. The inconvenience of using the spacebar to move forward all the time instead of using right and left arrow frequently causes frustration for the user. Also, the matrix operations in computing echelon form and reduced echelon form did not function very well on some Apple IIe computers.

However, Linear-Kit is considered to be the least expensive software of its kind which is available on the market at present, and it fulfills all the needs for a learner in elementary linear algebra. A student studying linear or matrix algebra will definitely find the benefits of using Linear-Kit for learning purposes or problem solving.

Title: Maple-Symbolic Computation for the Macintosh, Version 4.2.1
Authors: Bruce W. Char, Keith O. Geddes, Gaston H. Gonnet, Michael B. Monagan, Stephen M. Watt
Distributors: For the Macintosh: Brooks/Cole Publishing Co., 511 Forest Lodge Rd., Pacific Grove, CA 93950; Phone: (408) 373-0728

For all other systems: Waterloo Maple Software, 160 Columbia Street
West, Waterloo, Ontario, Canada, N2L 3L3
Phone: (519) 747-2373

Systems Requirements: Macintosh: Mac, Mac Plus, II, IIc, IIcx, SE, SE/30 or Portable with 1MB of memory; 10MB of free disk space.
Others: MS-DOS 80386, UNIX, VMS, and CMS.

Price: $395 (U.S.A.) - single user price

The Maple system was first conceived in 1980 at the University of Waterloo in an attempt to construct a CAS (Computer Algebra System) that would take advantage of advances in hardware and programming that had taken place since CASs like MACSYMA had been developed. It has gone through many changes since then. This review concerns version 4.2.1 run on a Macintosh Plus with 1 megabyte of RAM and a 20 megabyte hard disk, the minimum requirements needed to run this program. A more powerful computer would have given faster results, especially for plotting, and access to more mathematical functions.

The design of Maple which allows it to be so powerful for seemingly small memory requirements is that its kernel which performs arbitrary precision arithmetic, polynomial manipulation and Maple programming language interpretation loads into 500 KB of RAM. Other functions are contained in libraries that can be called with a Maple command as they are needed. The titles of some of these libraries will indicate the wide range of mathematical areas covered by Maple. They include Group Theory, Linear Algebra, Linear Optimization, Number Theory, Power Series, and Statistics. A recent Maple Fact Sheet also indicated that other libraries may be available on more powerful systems. These included Combinatorics, and Geometry and Projective Geometry. I also noticed from the Fact Sheet that functions I did not have were listed under libraries I did have in my version of Maple. For instance I was able to compute the eigenvalues of a matrix but not the eigenvectors while eigenvectors were mentioned on the Maple Fact Sheet.

Just the sheer number of topics listed above reflects on Maple favorably. But it is also relatively easy to use and should be no more intimidating to first year calculus students than, say, calculus itself! In fact it may help them to learn calculus as Maple will do just about all the algebra manipulations that students usually find so difficult and allow them to concentrate more on the concepts of calculus. Maple has been used to teach calculus at the University of Waterloo and other places. The experience from these trials has been used to develop a Maple library package called Student designed specifically for students of calculus. Among other things, Student does differentiation and indefinite integration as well as tangent lines and approximate integration using inscribed or circumscribed rectangles, the latter two with plotting.

Maple also has a programming language that should be easy for anyone to use that has programmed in a structured language. This feature helps to make Maple a tool of professional mathematicians and has been mentioned as an aid in solving some problems in mathematical journals.

The version that runs on the Macintosh makes full use of the Macintosh's easy editing features. It's possible to make one change in the middle of a long command, hit Enter and have Maple execute the revised command. One can also scroll back almost indefinitely and save Maple sessions for future use or editing.

I found the documentation received to be suitable. This included First Leaves for the Macintosh, a step-by-step 140-page tutorial covering the main features of Maple and its programming language and the Maple Reference Manual, a 400-page alphabetical listing of each command with information on its calling sequence, an explanation of the parameters, and several examples. Both Manuals were written by the 5 authors of Maple listed at the beginning of this review plus Benton L. Leong. There is also an adequate Help feature as part of the software. It is similar in structure to the Manual and it is more up to date.

Maple will do nearly all the symbolic algebra needed in high school algebra or first year calculus. In the following example we have Maple take the derivative of a fractional expression and then simplify it using the all-purpose Maple simplifying command "normal." Note that · is the primary Maple prompt and " means the last expression. The part in bold type indicates Maple output.

• expr2 := tan(x)/((x^2-1)^(1/2)); **expr2 :=** $\mathbf{\dfrac{\tan(x)}{(x^2-1)^{1/2}}}$

- diff(", x); $\frac{\sec(x)^2}{(x^2-1)^{1/2}} - \frac{\tan(x)\ x}{(x^2-1)^{3/2}}$

- normal("); $\frac{\sec(x)^2 x^2 - \sec(x)^2 - \tan(x)\ x}{(x^2-1)^{3/2}}$

In an interesting article (*Mathematics Teacher*, September 1990) entitled "Graphics Software Side by Side" Martha DiFazio compares several extent graphics packages, one criteria being how they graph certain difficult functions. One example was the graph of $y = \cos(40x)$ on the interval from 0 to 2π. Some packages were unable to cope with the many periods. Maple was able to draw it correctly although it took it a while. Another function discussed was $y = \frac{x-1}{(x-3)(x+2)}$ and the fact that many packages have difficulty displaying its asymptotes clearly. Maple did the acceptable job illustrated in figure 1(a). The "cleaner" graph in figure 1(b) was obtained by giving the domain as $(-\infty, \infty)$. Maple seemed to be able to handle all of the difficult graphs mentioned in the article.

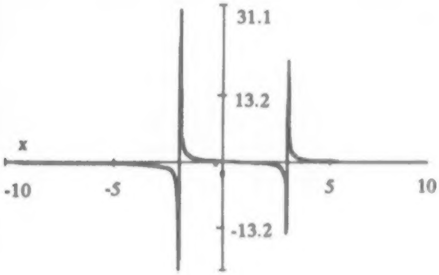

Figure 1(a)

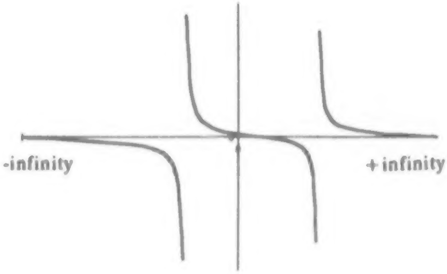

Figure 1(b)

There are some plotting features that I would have liked to have seen such as a simple ZOOM feature or a crosspiece that would have indicated the coordinates of the current position. An article by Barry Simon in the May 2, 1990 issue of *PC Magazine* has an interesting comparison of the CAS packages Derive, Macsyma, Maple and Mathematica. Maple does very well in the comparison but its weakest feature is said to be its graphical capabilities. However, information on the latest version, released after the article was written, promises improvements to Maple's graphical capabilities including 3-D plots.

In summary Maple is a relatively easy to use, powerful piece of mathematical software. The designers have 10 years of experience in its use as a mathematical and a teaching tool. Though there are features I have seen in packages dedicated to a particular mathematical area such as calculus or linear algebra that I would have liked to have seen in Maple, this was more than offset by the wide range of mathematics covered using a consistent command structure. Learning and using Maple would be particularly beneficial for students taking more than one mathematical area, for example math majors, and its powerful programming language make it a useful professional instrument as well as an educational one. Maple is a tool for the 21st century; we are fortunate to be able to use it in the 20th century!!

Reviewed by Doug Girvan, Red Deer College, Red Deer, AB, Canada

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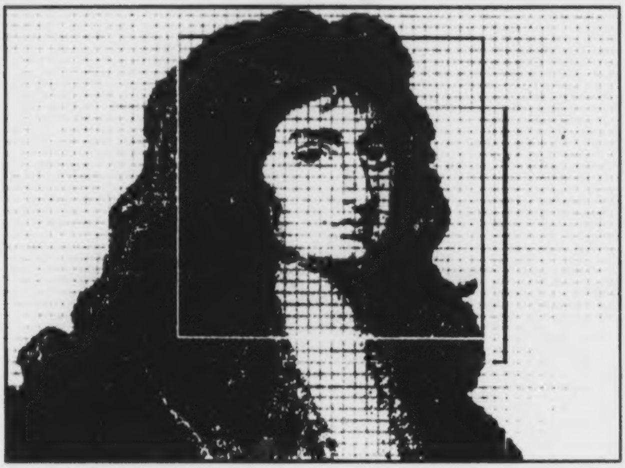

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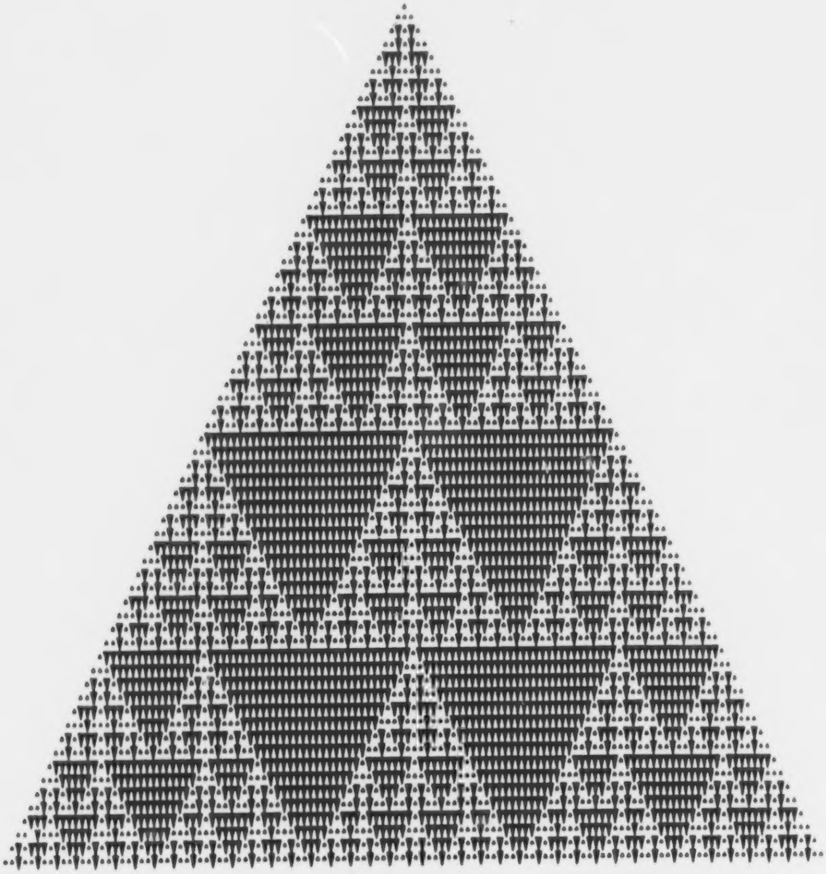

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Editor's Comments

A Period of Transition

My tenure as your editor will conclude with the publication of the Fall, 1992 issue. Your new editor will be "Chalkboard" columnist Joseph Browne. Although Joe will not officially become the editor until next November, all new manuscript submissions should now be sent to him at Onondaga Community College, Syracuse, NY 13215. He is currently working on the content for the Spring, 1993 issue.

Acknowledgements

Several changes occurred in our staff following the Seattle conference. Eleanor Young, our advertising manager retired and was replaced by Larry Gilligan. In addition, editorial panelists Travis Thompson and Martha Wood left the panel. Travis' term ended and Martha resigned to assume the chair of the Developmental Mathematics Committee. They have been replaced by August Zarcone and Jacqueline Thornberry. Sincere thank you's are extended to Eleanor, Travis, and Martha for their many years of service to *The Review*. It should also be noted that Nelson Rich, Gordon Hoagland, Brian Hickey, and Dennis Reissig were reappointed to three-year terms on the editorial panel. Finally, humorist Les Tanner's last column appears in this issue. Thanks for the chuckles.

New Opportunities - New Responsibilities

The ready availability of graphing utilities for classroom use offers us new opportunities as well as new responsibilities. Editorial Panelist Mike Davidson included the following comments in his review of an article dealing with the graphing of functions:

> If a student has access to graphing software, producing the graphs of functions becomes almost trivial, while explaining the intimate connections among the structures of the individual functions becomes paramount.... This article falls into the category of old techniques filtered through technology. It does not illuminate any higher level concepts or broad analytical methods.

These comments offer some sound advice to potential authors.

And Speaking of Opportunities

The Seventh International Congress on Mathematical Education will be held in Québec City on August 17-23, 1992. For information write to Congrès ICME-7 Congress, Université Laval, Québec, QC, Canada G1K 7P4.

Don Cohen, Editor

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ABOUT THE COVER

JE(3)*M*2

by John J. Edgell, Jr.
Southwest Texas State University, San Marcos, TX 78666

Dr. John J. Edgell, Jr., professor of mathematics at Southwest Texas State University, is currently on sabbatical. He first introduced the sequence of trigonal numerical arrays with JE(3) at the 9th AMATYC Annual Meeting. He has been associated with the Boy Scouts since 1948, earning the rank of Eagle and the volunteer honor of Silver Beaver. He rides 25 miles on a tandem bike with his son every morning.

JE(3)*M*2 is a trigonal graphical display of the distribution of even and odd coefficients of $(x^2 + x^1 + x^0)^{n-1}$ for any counting number, n, up to and including 64. A small black trigonal region represents an odd coefficient and a large black trigonal region represents an even coefficient. *JE*(3) represents an integral, J, extension, E, based upon 3. It is similar to the ancient Chinese trigonal array of numbers which is based upon 2 or the binomial expansion of $(x^1 + x^0)^{n-1}$. *JE*(3)*M*2 merely emphasizes the coefficients of $(x^2 + x^1 + x^0)^{n-1}$ which are a multiple, M, of 2. Notice that there appears to be a symmetrical and cyclic distribution of the even and odd coefficients.

The graph may give rise to some interesting elementary conjectures about the distribution of the even (or odd) coefficients of $(x^2 + x^1 + x^0)^{n-1}$. For instance, one may conjecture that the even (or odd) coefficients are symmetrically distributed about a particular term of the expansion. Furthermore, one may conjecture that the cyclic distribution of the even coefficients occur as symmetric trigonal arrays, pairwise in levels, and similar to the same arithmetic sequence, (1, 3, 5,...) reflected in *JE*(3) but in reverse order. One may even predict the cycles of the symmetric distributions, perhaps in powers of 2. Also, the graph gives the impression that $(x^2 + x^1 + x^0)^{1-1}$ and $(x^2 + x^1 + x^0)^{2-1}$ are the only expansions of $(x^2 + x^1 + x^0)^{n-1}$ that do not have even coefficients. As one studies the graph more intently, other elementary conjectures seem to readily occur.

Acknowledgement

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VIEWPOINT

A Time for Change in Remedial Mathematics

by Edward D. Laughbaum
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Columbus, OH 43215

Ed has taught at Columbus State since 1971. He was a presenter at the International Conference on Technology and Education in Brussels, Belgium in 1990 and is currently President-Elect of OhioMATYC.

As dialogue on the revolution of mathematics education continues, the time for action is upon us. Demana and Waits (1990) have already taken action in college algebra and trigonometry, and now is the time to act on remedial mathematics education as well. Professional organizations such as the Mathematical Sciences Eduction Board of the National Research Council, AMATYC, NCTM, and MAA have all suggested changes in college mathematics curriculum in the publications listed on the reference page. The seed has been planted. New instructional methods need to be nurtured.

The Current Situation

Lynn Arthur Steen (1991) maintains that most mathematics remediation is a failure. As currently taught, our elementary mathematics courses are so burdened with symbolic manipulations that it is difficult for instructors to spend adequate time on fundamental concepts (Karian, 1990). For example, during the 1989 summer quarter, all 62 Intermediate Algebra students at Columbus State Community College were tested on most of the traditionally taught skills at the remedial level. The study confirms Steen's statement; it shows that 48% of the students couldn't factor polynomials at the beginning of the course, nor could they factor the same polynomials at the end of the course. They had been taught factoring in a beginning-level course, and it was reviewed in the intermediate course. Only 17% of the students learned to factor during the course. For manipulation skills with rational expressions, 39% couldn't manipulate them when they started the course, nor could they manipulate fractions at the end of the course. Even worse, seventy percent failed to learn how to solve radical and quadratic equations during the course! With failures like these, remedial instruction, both in method and content, needs to be re-evaluated; we must reconsider what content is appropriate and what is not (Davis, 1989).

Can we be successful at remediation? Not by teaching remedial courses using a standard approach which repeats high school methods "faster" and "louder" (Davis, 1989). One problem may be that curricula, texts, tests, and teaching habits are all products of the precomputer age, although the students are not (National Research council, 1989). To compound the problem, teaching mathematicians continue to discuss how mathematics should be taught, while many high school teachers have already begun to implement the recommendations in the *STANDARDS* published by NCTM (1989). Teaching *STANDARDS* mathematics in high school and traditional mathematics in college remedial courses could prove disastrous to remedial mathematics education.

A Proposed Direction

To change the current pattern of instruction, I propose that teaching methods be changed to support implementation of the graphing calculator into the remedial sequence. We must take the opportunity to incorporate visualization into the instructional process at the remedial level as well as at other levels (Davis, 1989). Technology has given mathematics educators the opportunity to restructure the basic remedial courses. This restructuring will promote the development of methods that allow for the teaching of concepts which emphasize understanding of mathematics as well as methods that develop new thinking skills. Restructuring should make our students better mathematical thinkers rather than simply symbol manipulators (Ralston, 1989a). To do this, however, we must be willing to decrease the time devoted to manipulation skills (Ralston, 1989a).

The use of graphing calculators in the mathematics classroom makes many traditional assumptions about student readiness and topic placement obsolete (Foley, 1990). For example, the concept of function can be included early in a beginning level course so that the linear function, absolute value function, square root function, and quadratic function can be *introduced* in the same course. Thus, instructors can use these functions in the **conceptual** development of factoring, arithmetic operations with polynomials, simplification of radicals, solving equations, solving inequalities, and error analysis. In addition, the study of these functions also permits modeling of many real-world situations at the remedial level. When algebra is approached graphically, the introduction to analytical geometry becomes a natural extension of the study of functions and should be included at the beginning level.

Do we really want our remedial mathematics students to memorize how to solve each type of equation they encounter or to memorize how to factor each polynomial type? Shouldn't they instead be taught the relationship between zeros of functions and real solutions to any equation, the relationship between the zeros of polynomial functions and factors of polynomials, and the relationship between zeros of functions and solutions to all types of inequalities? The graphing calculator can be used to find zeros which, in turn, can be used to solve equations and inequalities or factor polynomials. In the process of finding zeros, a discussion of error can and must take place. Our students need to be aware that many real-world problems do not have exact solutions, and if solutions are not exact, they can learn how to control the error in the solution.

If the students are armed with a graphing calculator, they can solve nearly any equation encountered in a two-year college curriculum using the same graphical approach. The graphing calculator gives students visual and numerical information on how they can solve an equation. If we teach the geometry of equations – the visualization of equations – the students will know the following even before solving an equation: (a) the number of possible real solutions; (b) whether the solutions are positive or negative; (c) what changes in the equation would cause more or fewer solutions; (d) which solutions apply to the problem situation; and (e) how to find any desired solution.

Just as Demana and Waits (1990) have taken a few traditional calculus topics and developed them in college algebra, selected topics from college algebra can and should be developed graphically in intermediate algebra and beginning algebra. For example, a complete analysis of the linear and quadratic functions should be included in intermediate algebra with an introduction in beginning algebra. The absolute value and square root functions should be introduced in remedial mathematics, with a thorough investigation taking place in college algebra.

Additions to remedial mathematics necessitates deletions from remedial mathematics. For example, the manipulation skills of operations with polynomials, radical expressions, exponential expressions, and fractions can be shortened considerably by lowering the level of sophistication. Also, we do not need to solve excessively complicated equations by algebraic methods when they can be solved graphically without simplification. When remedial students are taught relationships between zeros of functions, roots of equations, solutions to inequalities, and factors of polynomials, the collective time devoted to these and related topics is much less than the time required to teach these topics individually from an algebraic perspective.

Conclusion

Research suggests that too much drill results in having the mathematical message hidden by the symbol manipulation (Ralston, 1989a). Developing skills which students know they will never need to use is turning them off to mathematics (Ralston, 1989a). In many cases this is precisely what we are doing at the remedial level – teaching skills that many students will never use – and the students are responding by leaving the study of mathematics at the rate of 50% per year (National Research Council, 1989).

The thrust of our remedial courses should be to teach material as a coherent study of interrelated concepts rather than as a sequential collection of isolated topics (Davis, 1989). Teaching isolated topics is time consuming and not beneficial to understanding mathematics. The use of calculators for computation and graphing will force students (and faculty) to study material at a more sophisticated level (Davis, 1989).

The Mathematical Sciences Education Board of the National Research Council (1990) maintains, "... algebraic skills such as polynomial arithmetic no longer serve a compelling purpose." Our currently taught manipulative skills are not being learned well enough to declare remediation a success, and current thinking is that these skills are not crucial to success in mathematics. We shouldn't continue to teach these topics when students can succeed in mathematics without demonstrating proficiency at these skills.

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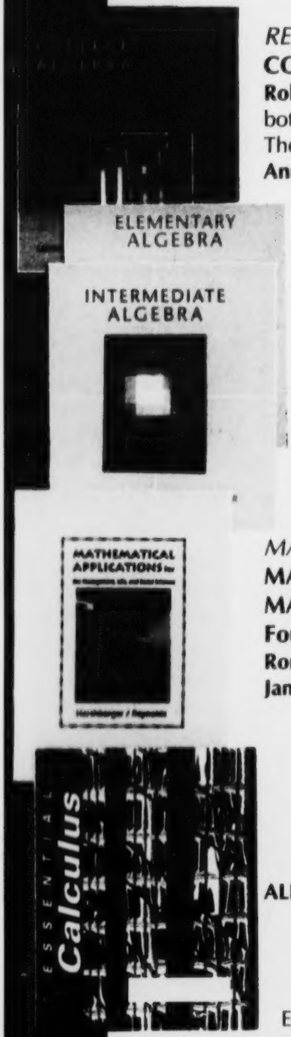

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MATHEMATICAL EXPOSITION

An Application of Discrete Mathematics to Coding Theory

by L. Joyce Donohoe
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Joyce received her BS and MA from Georgia State University. Her industrial background includes working with the CAT scanner and NMR for General Electric Medical Systems. She has taught at Southwest Texas State since 1987.

The rapid spread of computers in our society during the past decade has had a serious impact on college mathematics. Mathematical needs for many students have changed at a fundamental level and the mathematical community has had to react quickly to the demand for change.

Recursion is the central theme of the needed mathematics and many branches of mathematics are brought together under this umbrella. However, because of the digital nature of computer algorithms and the natural alignment of calculus with continuity, calculus topics are rarely referenced. What is needed is a mathematics that deals with sets of discrete objects together with the logical and algebraic relations between these discrete objects. This study is known as "discrete mathematics." (The reader is advised to consult the Mathematical Association of America, 1986, *Report: Committee on Discrete Mathematics in the First Two Years* for their recommendations on the topics that should be included in a discrete mathematics course.)

The purpose of this paper is to show how a specific application can be used to introduce students to several topics in discrete mathematics. The application is a well-known public-key cryptosystem. For some, the ultimate goal of such a system is to publish an encryption key along with the usual name, address, and telephone number for each entry in a computerized telephone directory. If such a directory existed, everyone would have a publicly known encryption key; namely, a pair of integers (N, e), that the computer would use to encode a message to be transmitted. In addition, each person would have an individual secret key (N, d) that could be used to decode the message. It should be noted that the example uses small numbers so that the focus is on how the encryption/decryption process works. The directory should be thought of as being computer-accessible because in a practical scenario one of the integers would involve at least 200 digits.

Diffie and Hellman (1976) outline a general scheme for such a public-key cryptosystem to work. They observe that if such a system could be devised, where we let M stand for a message to be transmitted, P stand for the public key function that encrypts the message, and S stand for the secret key that decrypts the encoded message, then the four conditions for such a system to be successful are:

1. $S(P(M)) = M$ for each M, so that we can recover the message.
2. All (S, P) pairs must be distinct so that individuals would only be able to decode messages intended for them.
3. Deriving the secret key from the public key must be at least as difficult as decoding the encrypted message. Otherwise, there is no advantage for having the secret key.
4. Both S and P are easy to compute, meaning that the time complexity for the algorithm would be reasonable.

Rivest, Shamir, and Adleman (1978) discovered a method to find S and P. Their system is known as the RSA system. Here the RSA Algorithm is documented after stepping through an application. The algorithm is easier to understand once the application has been seen.

Some Needed Theory

Theorem 1. *Corollary to Euler's Theorem (A.D. 1700s.)* If p and q are distinct primes, $n = pq$, and $\gcd(a, n) = 1$, then $a^{(p-1)(q-1)} \equiv 1 \pmod{n}$.

For example, if we let $p = 97, q = 103$ ($n = pq = 9991$), and $a = 7269$, where we know $\gcd(7269,9991) = 1$, then $7269^{9792} \equiv 1 \pmod{9991}$.

Remark: Since $(11)(4451) \equiv 1 \pmod{9792}$, $(11)(4451) = 1 + k(9792)$ for some integer k and 11 is the multiplicative inverse of 4451 (mod 9792). Furthermore,

$$\begin{aligned} 7269^{(11)(4451)} &= 7269^{1+k(9792)} \\ &= (7269)\left(7269^{9792}\right)^{k} \\ &\equiv (7269)(1(\text{mod } 9991)) \\ &\equiv 7269 \quad (\text{mod } 9991)\,. \end{aligned}$$

Theorem 2. *Euclid's Algorithm (300s B.C.)* Given positive integers b and c, the last nonzero remainder produced by the Euclidean algorithm equals $\gcd(b, c)$.

For example, let $b = 7269$, $c = 9991$, then $\gcd(7269, 9991) = 1$ where 1 is the last nonzero remainder produced by the following application of this algorithm:

$$\begin{aligned} 9991 &= 7269 + 2722 \\ 7269 &= (2)(2722) + 1825 \\ 2722 &= (1)(1825) + 897 \\ 1825 &= (2)(897) + 31 \end{aligned} \qquad \begin{aligned} 897 &= (28)(31) + 29 \\ 31 &= (1)(29) + 2 \\ 29 &= (14)(2) + 1 \\ 2 &= (2)(1) + 0 \end{aligned}$$

The proofs for the theorems are found in most standard number theory texts. For example, see Stark (1978).

The Application

We are now ready for the application. Albertson and Hutchinson (1988) consider a problem in which Bob wants to send a message to Alice but keep the message a secret from Eve. Suppose Bob's message is "HELLO." "HELLO" is 7269767679 in ASCII. We separate 7269767679 into three blocks: 7269, 7676, 7932 each with four digits. Notice 32 represents a blank to fill out the last block.

Begin by finding an integer N that is (*i*) larger than the biggest block and (*ii*) a product of two primes. We choose $N = 9991 = (97)(103)$. Then we choose e such that $1 < e < (97)(103)$ and e is relatively prime to $(97-1)(103-1)$. The significance of naming this integer "e" is that it will be an exponent. We let $e = 11$ and we let $d = 4451$. Recall from the remark following Theorem 1 that d and e are multiplicate inverses (mod 9792).

Without loss of generality, we can restrict our illustration to transmitting and decoding only the first block. Let $M = 7269 =$ "HE." Check to see that $\gcd(7269,9991) = 1$. If $\gcd(7269,9991) \neq 1$, then we have no guarantee that $7269^{(96)(102)} \equiv 1 \pmod{9991}$, which we require to decode our block (see Theorem 1).

Observe the following recursion relationship:

$$\begin{array}{ll}
(M)^1 = M & (M)^1 = 7269 \pmod{9991} \\
(M)^2 = ((M)^1)^2 & (M)^2 = 5953 \pmod{9991} \\
(M)^4 = ((M)^2)^2 & (M)^8 = 7433 \pmod{9991} \\
(M)^8 = ((M)^4)^2 & \\
M^{11} = M^8\,M^2\,M^1 &
\end{array}$$

Since $M^{11} = M^8M^2M^1$, we have the following:

$$\begin{aligned}
M^{11} &= M^8\,M^2\,M^1 \quad \pmod{9991} \\
&= 7433 \cdot 5953 \cdot 7269 \pmod{9991} \\
&= 9425 \pmod{9991} \\
&= R\,,
\end{aligned}$$

where R represents the encryption of M.

The use of the powers of 2 emphasizes the recursive way a computer program would be designed to save the results of former multiplications for use in computing future multiplications, thus saving time. There are many clever algorithms available for multiplying integers based on representing the integers in base 2 and using divide-and-conquer techniques. For example, see Knuth (1981). To recover M from R, recall that 4451 is the multiplicative inverse of 11 (mod (96)(102)) which implies that $(7269^{11})^{4451} \equiv 7269 \pmod{9991}$. To show R^{4451} is equal to 7269 (mod 9991) it suffices to show that $7269^{11} \equiv 9425 \pmod{9991}$.

We show this as follows:

$$\begin{aligned} 7269 &\equiv 7269 \pmod{9991} \\ 7269^2 &\equiv 5953 \pmod{9991} \\ 7269^8 &\equiv 7433 \pmod{9991} \end{aligned}$$

$$\begin{aligned} 7269^{11} &= 7269^{8+2+1} \\ &= 7269^8 \cdot 7269^2 \cdot 7269 \\ &\equiv 7433 \cdot 5953 \cdot 7269 \pmod{9991} \\ &\equiv 7433 \cdot 1336 \pmod{9991} \\ &\equiv 9930488 \pmod{9991} \\ &\equiv 9425 \pmod{9991} \end{aligned}$$

The same algorithm for encryption would be used for decryption by the person who receives the message because they would not know 7269. They would know only R.

$$R^{4451} = 9425^{4096+256+64+32+2+1},$$

where the emphasis is on powers of 2.

Recall that Bob wants to send the message 7269 to Alice and keep that message a secret from Eve. All three individuals know the public key (N, e) where $e = 11$, $N = 9991$, but Alice alone knows the secret key (N, d), $d = 4451$. Eve does not know d. If Eve intercepts the message, can she decode it from her knowledge of the public key? The answer is "yes"; however, if her approach is to take the 11*th* root of R she finds the answer is 2.2977... which is of little help. The reason that this approach will not work is that Bob took the 11*th* power of 7269 mod 9991 (not the 11*th* power of 7269) and so Eve must take the 11*th* root of 7269 mod 9991 (not the 11*th* root of 7269).

The straightforward approach Eve would use to decode the message is to calculate 1^{11} mod 9991, 2^{11} mod 9991, etc., which is indeed time consuming. However, by looking at an ASCII table, Eve can see that M must be of the form 3232 or $uvwz$ where $65 \leq uv, wz \leq 90$ or $uv = 32$ where $65 \leq wz \leq 90$ or $wz = 32$ where $65 \leq uv \leq 90$. So Eve need calculate only the eleventh power of these 729 possibilities.

Assuming Eve's PC can perform 17,800 single digit multiplications and divisions, it would take Eve about 10 minutes to find M, since $i^{11} = i^8 i^2 i^1$ for any integer i, would require five multiplications (i^2, i^4, i^8 plus two more to combine into i^{11}), five divisions for reducing modulo 9991 and five more to recombine, making a total of 10,935 operations.

By using all the ASCII characters and separating the ASCII message into blocks larger than four decimal digits, Albertson and Hutchinson show a systematic search may require $(\log 10^{B+1})(10^{B+1})$ operations where B is the decimal digit length of the block. This would be an exponential amount of work for Eve. If $B \geq 6$, it would take Eve's PC at least a month.

The RSA Algorithm

The RSA Algorithm is as follows:

Step 1. Calculations performed by receiver.

- (a) Select primes p and q, and let $N = pq$.
- (b) Select a random number e between 1 and N, such that $\gcd(e, (p-1)(q-1)) = 1$.
- (c) Find d such that $ed \equiv 1 \pmod{(p-1)(q-1)}$ where $0 < d < (p-1)(q-1)$.
- (d) Publish the pair (N, e) in the telephone directory.

Step 2. Encryption

- (a) Represent the message in ASCII (a well-known computer character code).
- (b) Choose integer $B <$ the number of decimal digits in N.
- (c) Break the ASCII coded message into blocks $M_1, M_2, \ldots, M_k$, each of length B with an ASCII blank to fill out the last block if necessary.
- (d) For $i = 1,2, \ldots, k$ check to see that gcd $(M_i, N) = 1$; if not the code is "broken"; return to step 1.
- (e) For $i = 1,2, \ldots, k$ let $R_i \equiv (M_i)^e \pmod N$ where $0 < R_i < N$.
- (f) Transmit $R_1, R_2, \ldots, R_k$.

Step 3. Decryption

- (a) For $i = 1,2, \ldots, k$ calculate $M_i \equiv (R_i)^d \pmod N$ where $0 < R_i < N$.
- (b) For $i = 1,2, \ldots, k$ translate M_i from ASCII code.

Summary

I use an application such as this one in the classroom to introduce the student to discrete mathematical topics that are of direct benefit in the field of computer science. After working through such an application, the student seems to understand the need to master the theorems presented in the course. I believe it is important that the computer science student learn not only the mathematical tools available for use in the field but also have an appreciation for the theory that undergirds the techniques.

After going through the application presented in this article and confirming that the discrete mathematical techniques used are well understood, one can move on to constructing a scheme which is based on the same principles but which is much more difficult to subject to a systematic search. In practice, three large ($\approx$ 100 digits long) random prime numbers are generated. The largest is set to e and the other two are p and q. With the advent of special purpose parallel processing machines, the code can be broken must faster than when it was first implemented; however, in certain applications, such as those used by the government involving satellite transmission, the key is changed every few seconds which makes code-breaking impractical. Also in practice the message is broken up into numbers less than N by using a binary string corresponding to the character encoding and taking $\log_2 N$ bits at a time.

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S2122

Symmetry in Irregular Convex Polygons

by Thomas W. Shilgalis
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Tom has been at Illinois State University since 1967. His special interests are geometry and computer simulations in probability, along with the state high school contest conducted by the Illinois Council of Teachers of Mathematics. He received a PhD in mathematics education from the University of Illinois in Urbana in 1971.

Symmetry is a topic frequently encountered in geometry classes, and regular polygons are often used to illustrate the concepts of bilateral and rotational symmetry. A regular polygon of n sides has $2n$ symmetries, i.e., transformations which leave the polygon invariant. These consist of reflections across the n symmetry lines of the polygon and n rotations about the center of the polygon. Consecutive symmetry lines form an angle of $180°/n$. The rotation angles are even multiples of $180°/n$, from 2 through $2n$. For example, the equilateral triangle has three lines of symmetry. The angle between consecutive symmetry lines is 60°, and the rotations are through 120°, 240° and 360°. The $2n$ symmetries of a regular n-gon constitute the *dihedral group of order 2n*. This is a noncommutative group. As such, it provides an opportunity for teachers to illustrate noncommutativity in a geometric context.

While investigating the topic of symmetry in a course for secondary teachers, the author and his students strayed a bit to consider bilateral symmetry in *irregular* polygons. An interesting question which arose was: What is the maximum number of lines of symmetry in an irregular polygon of n sides? The results are expressed in the following theorem:

Theorem. Let $n \geq 3$ be an integer, and let d be the largest divisor of n less than n.

(a) There exists an irregular convex polygon of n sides having d lines of symmetry.
(b) There does not exist an irregular convex n-gon with more than d lines of symmetry.

Figure 1 shows irregular polygons of 6, 7, 8, and 9 sides, having 3, 1, 4, and 3 symmetry lines, respectively. We note that symmetry lines are of three types:

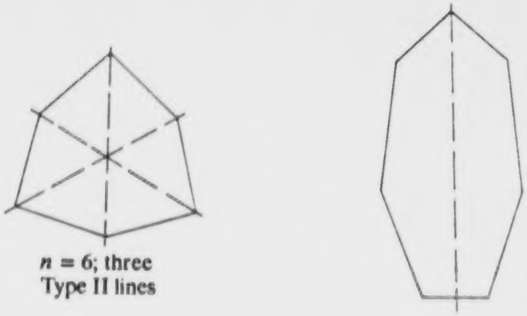
Type I: a line passing through a vertex and the midpoint of the opposite side when the number of sides is odd.

Type II: a line passing through two opposite vertices when the number of sides is even.

Type III: a line passing through the midpoints of two opposite sides when the number of sides is even.

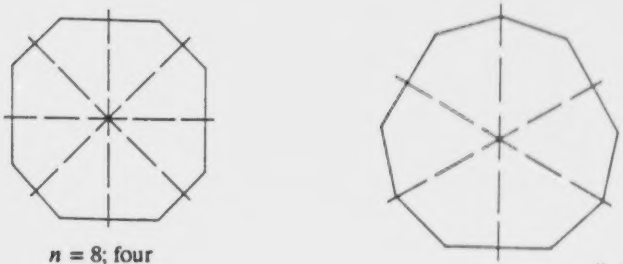
Proof. A proof of the given theorem is as follows:

Part (a): To prove that there exists an irregular convex polygon of n sides having d lines of symmetry, we consider two cases.



$n = 6$; three Type II lines

$n = 7$; one Type I line



$n = 8$; four Type III lines

$n = 9$; three Type I lines

Figure 1

Case 1: When n is odd, there are two subcases.

1(i): If n is prime, $d = 1$. Choose a regular n-gon and select one of its n symmetry lines, L. Compress the regular polygon toward L by, say, a factor of one-half. The resulting n-gon has just one symmetry line, L. The heptagon in Figure 1 was obtained in this manner.

1(ii): If n is odd and composite, let d be its largest proper divisor. We begin with a regular d-gon and build an irregular n-gon from it by retaining the symmetry lines of the d-gon as symmetry lines of the n-gon.

Let C be the circle circumscribed about the regular d-gon. (See figure 2, where $n = 35$ and $d = 7$.) The odd integer n/d is at least as large as 3 since $d < n$. We describe a procedure to insert $(n/d) - 1$ vertices on C between each pair of consecutive vertices of the d-gon, thus replacing each side of the regular d-gon with n/d segments. The union of these n new segments will form an n-gon which is symmetric about each of the lines of symmetry of the original regular d-gon.

If V_1 and V_2 are consecutive vertices of the d-gon, divide the arc $\widehat{V_1V_2}$ into n/d subarcs $\widehat{V_1A_{1,1}}$, $\widehat{A_{1,1}A_{1,2}}$, ..., $\widehat{A_{1,(n/d)-1}V_2}$, where the first and last of these arcs have measure x and the rest have measure y. The measures x and y are chosen to satisfy $0 < y < x$ and $2x + \left[(n/d) - 2\right]y = 360°/d$. The procedure retains the symmetry with respect to the perpendicular bisector of segment $\overline{V_1V_2}$, which is one of the symmetry lines. The inequality of x and y guarantees that the polygonal path $V_1A_{1,1}$, ..., $A_{1,(n/d)-1}V_2$ is not symmetric with respect to any other line. This procedure is then copied between each pair V_i and V_{i+1}, $i = 1, 2, ..., n-1$, to produce an irregular

n-gon whose symmetry lines are those of the original d-gon. Figure 2 shows, for only one side, the case $n=35$, $d=7$, where x and y are chosen to be 15° and $50/7^\circ$, respectively.

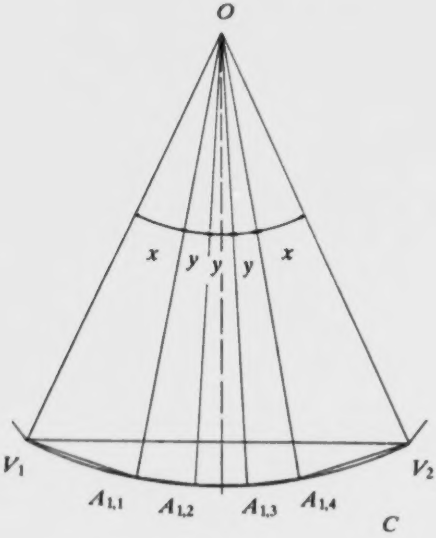


Figure 2

Case 2: Let n be even. When $n=4$ the nonsquare rhombus, with its two Type II symmetry lines, and the nonsquare rectangle, with its two Type III symmetry lines, suffice. When n is even and greater than 4, we have two different methods to construct an irregular n-gon with $d=n/2$ symmetry lines. We note that $d > 2$.

The first way involves starting with a regular d-gon and replacing each corner with a new segment to form a $2d$-gon, i.e., an n-gon. The new segments are chosen so that their endpoints are equidistant from the corners that are being replaced. This amounts to clipping off congruent isosceles triangles from a paper model of a regular d-gon. The octagon in Figure 1 was constructed in this manner. To ensure that the resulting n-gon is not regular, we choose the legs of the "clipped-off" isosceles triangles to be one-fourth as long as the sides of the d-gon. The d symmetry lines of the n-gon so constructed are all of Type III and the n-gon is equiangular but not equilateral. There are two different side lengths which alternate around the n-gon.

The second way involves starting with a regular n-gon and bringing alternate vertices in toward the center by equal amounts, while retaining convexity. The Type II symmetry lines of the regular n-gon, of which there are d, are symmetry lines of the new n-gon, but the type III symmetry lines of the regular n-gon are not symmetry lines of the new n-gon, which is equilateral but not equiangular. There are two different central angle measures which alternate around the center of the n-gon. The hexagon in Figure 1 was constructed in this manner.

This completes the proof of the existence part of the theorem.

Part (b): To prove that there does not exist an irregular convex n-gon with more than d lines of symmetry, we again consider two cases.

Case 1: Let n be odd. Suppose that the n-gon has exactly k symmetry lines, where $k > 1$. We show that k must divide n, from which it will follow that, if $k > d$, where d is the largest proper divisor of n, then $k = n$; in that event the n-gon must be regular.

The author (see Shilgalis, 1972) has shown that if a plane figure has exactly $k > 2$ symmetry lines, then these lines are concurrent at a point O and the angle between each pair of consecutive symmetry lines is $180°/k$. Three of the four polygons in Figure 1 illustrate this property of symmetry lines.

Let O be the center of an n-gon having exactly k symmetry lines. Let T be the clockwise rotation of $360°/k$ about O. We note that T, which is expressible as the product of two reflections across consecutive symmetry lines, leaves the n-gon invariant since each of the reflections which comprise T does so. We choose L_1 and L_2 as a particular pair of consecutive symmetry lines, ordering their labeling so that $T(L_1) = L_2$. Denote by V_1 the vertex lying on L_1. (Since n is odd each symmetry line is of Type I.) Let $T(V_1) = V_2$. V_2 is a vertex since it lies on the images of two sides which meet at V_1. Note that L_2 and L_1 must be different, for otherwise the symmetry line L_1 would contain two vertices, V_1 and V_2, which is impossible since n is odd. It follows that $k > 2$. Similarly, $T^2(V_1), \dots, T^{k-1}(V_1)$ are all vertices, and $T^k(V_1) = V_1$. Denote $T^{r-1}(V_1)$ by V_r, for $r = 2, \dots, k$. Note that T generates the rotational subgroup of the symmetry group of the n-gon and no other generator of this subgroup has a smaller rotation angle.

Now suppose that $m > 0$ vertices lie in the interior of angle V_1OV_2. Call these $V_{1,1}, \dots, V_{1,m}$. Since L_2 is a symmetry line, the images $V_{2,1}, \dots, V_{2,m}$ of $V_{1,1}, \dots, V_{1,m}$ under reflection across L_2 lie in the interior of angle V_2OV_3, and so on. The interiors of angles $V_1OV_2, \dots, V_kOV_1$ do not overlap, so the n vertices of the n-gon are separated into two sets

$$\{V_1, \dots, V_k\} \text{ and } \{V_{1,1}, \dots, V_{1,m}, V_{2,1}, \dots, V_{2,m}, \dots, V_{k,1}, \dots, V_{k,m}\}.$$

The first set consist of vertices which lie on the k symmetry lines. The second set consists of k subsets, each with m vertices, which do not lie on the k symmetry lines. Thus $n = k + km$. It follows that k divides n. Now the supposition is that $k > 1$, so if n is prime, k must be equal to n, and the n-gon is regular. If n is not prime, then k is a proper divisor of n. This completes the proof in the case where n is odd.

Case 2: Let n be even. Let $n = 2d$ and suppose, by way of contradiction, that the irregular n-gon has k lines of symmetry, where $k > d$. No symmetry line is of Type I when n is even. Since there are d pairs of opposite sides and d pairs of opposite vertices, at least one Type II symmetry line must be present among the k symmetry lines. The argument in Case 1 only used the oddness of n to ensure the existence of at least two symmetry lines which passed through vertices. So if, in the present case, there are at least two symmetry lines passing through vertices, the remainder of the argument in Case 1 applies to show that k is a divisor of n, and consequently, $k = n$ and the n-gon is regular, which is a contradiction.

But if, in the present case, there is only one symmetry line passing through a vertex, then there are d symmetry lines of Type III (passing through the midpoints of pairs of

opposite sides). As noted in the existence part of the proof, the presence of *d* symmetry lines of Type III when *n* is even forces the *n*-gon to be equiangular and have just two different side lengths, occurring alternately. An additional symmetry line of Type II (passing through a vertex) forces these two side lengths to be the same. Thus, the presence of more than *d* symmetry lines implies that the *n*-gon is regular, a contradiction. Thus, when $n = 2d$, an irregular *n*-gon cannot have more than *d* lines of symmetry.

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Patterns Related to the Decimal Expansions of 2^{-n}

by Andrew Cusumano
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Exploring for numerical patterns can be a fascinating and challenging activity for mathematics students at all ability levels. The unsophisticated student can "do mathematics" by generalizing from patterns of observations. The conjectures can be verified (in a non-rigorous manner) by extending the patterns beyond the original number of cases. In addition, patterns offer the more sophisticated student the opportunity to practice rigorous deductive reasoning. Such students can verify conjectures using logical proofs.

Table 1. Decimal expansions of 2^{-n}

n	2^{-n}
1	.5
2	.25
3	.125
4	.0625
5	.03125
6	.015625
7	.0078125
8	.00390625
9	.001953125
10	.0009765625
11	.00048828125
12	.000244140625
13	.0001220703125
14	.00006103515625
15	.000030517578125
16	.0000152587890625
17	.00000762939453125
18	.000003814697265625
19	.0000019073486328125
20	.00000095367431640625
21	.000000476837158203125
22	.0000002384185791015625
23	.00000011920928955078125
24	.000000059604644775390625
25	.0000000298023223876953125
26	.00000001490116119384765625
27	.000000007450580596923828125
28	.0000000037252902984619140625
29	.00000000186264514923095703125
30	.000000000931322574615478515625

This article describes some numerical patterns that exist in the decimal expansions of the negative powers of 2. Before reading the rest of the article, the reader is invited to carefully study Table 1 and make a list of numerical patterns. How many conjectures can you make?

Row Patterns

Probably the first pattern that most of you recognized was that the number of digits in each decimal expansion is the same as the row number.

Conjecture 1. The number of digits in the decimal expansion of 2^{-n} is n.

$$\begin{aligned} 2^{-1} &= .5 \\ 2^{-2} &= .25 \\ &\vdots \\ 2^{-n} &= \underbrace{.a_1a_2\ldots a_{n-1}5}_{n \text{ digits}} \end{aligned}$$

Proof. Each 5 at the end of a decimal expansion is understood to be followed by a zero. Since division by 2 into a number ending in 5 always results in a carry, each succeeding expansion has an additional digit; namely, 5.

Notice that the last five digits in the fifth expansion; namely, 03125 occur as the last five digits in the expansions in rows 13, 21, and 29. In general, it appears that the last n digits in the decimal expansions whose row numbers are congruent to n mod 2^{n-2} are the same. Furthermore, in a more general sense, you do not have to consider all n digits to the right of the decimal point in the n^{th} expansion.

Conjecture 2. The last n' places ($2 \leq n' \leq n$) of the decimal expansions whose row numbers are congruent to n mod $2^{n'-2}$ agree in their last n' decimal places.

For example, the last $n' = 4$ digits in the fifth row also appear at the ends of rows 9, 13, and 17. Each is congruent to 5 mod 4.

n	
5	$.0\underline{3125}$
6	.015625
7	.0078125
8	.00390625
9	$.00195\underline{3125}$

You may use this apparent pattern to determine the exact digit near the end of a decimal expansion of 2^{-n} for large n. For example, what is the 26th digit (the fifth from the end) in the decimal expansion of 2^{-30}? Since 30 is congruent to 6 mod 2^{5-2}, the fifth to the last digit in the 30th row is the same as the fifth to the last digit in the sixth row; namely, 1.

Proof. The following proof is based on the observation that two integers will agree in their last k digits if, and only if, the difference contains a factor of 10^k. For example, $365421 - 4421 = 361000 = 361\times10^3$.

Since the decimal expansion of 2^{-n} has exactly n digits, $10^n(2^{-n}) = 5^n(2^n)(2^{-n}) = 5^n$ is an integer with exactly the same sequence of digits as 2^{-n}. Therefore,

$$D = \begin{bmatrix}\text{the integer in row } n+2^{k-2} \\ \text{with the same sequence of} \\ \text{digits as those of } 2^{-\left(n+2^{k-2}\right)}\end{bmatrix} - \begin{bmatrix}\text{the integer in row } n \text{ with} \\ \text{the same sequence of digits} \\ \text{as those of } 2^{-n}\end{bmatrix}$$

$$= 10^{n+2^{k-2}}\left(\frac{1}{2^{n+2^{k-2}}}\right) - 10^n\left(\frac{1}{2^n}\right)$$

$$= 5^{n+2^{k-2}} - 5^n$$

$$= 5^n\left(5^{2^{k-2}}-1\right)$$

$$= 5^n\underbrace{\left(5^{2^{k-3}}+1\right)\left(5^{2^{k-4}}+1\right)\left(5^{2^{k-5}}+1\right)\ldots\left(5^{2^1}+1\right)\left(5^{2^0}+1\right)}_{(k-2)\text{ factors each divisible by 2, but not by 4.}}\underbrace{\left(5^{2^0}-1\right)}_{4=2^2} \qquad (1)$$

Note on line (1) that since $5\equiv1 \bmod 4$, $5^x\equiv1 \bmod 4$. Since 5^x-1 and 5^x+1 are consecutive even numbers with 5^x-1 divisible by 4, it follows that for all x, 5^x+1 is not divisible by 4. Therefore, line (1) contains a total of exactly k factors of 2. Continuing,

$$\begin{aligned} D &= 5^n\left(5^{2^{k-2}}-1\right) = 5^n\cdot2^k\cdot Q \quad (Q\text{ odd}) \\ &= 10^k\left(5^{n-k}\cdot Q\right) \qquad (2)\end{aligned}$$

Since $k < n$, then $n-k > 0$ and k is the maximum power of 10 contained in D. Therefore, the decimal expansion of any two negative powers of 2 separated in the table by 2^{k-2} rows, agree in their last k places.

Diagonal and Complement Patterns

The complement of a digit is that number plus or minus five, whichever results in a number from zero to nine. For example, the complement of 2 is 7 and the complement of 6 is 1.

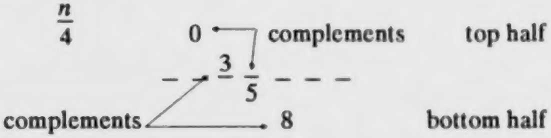
If we define the n^{th} diagonal in Table 1 to be the ordered sequence of digits in the n^{th} to the last positions of the decimal expansions, we see that the digits on the various diagonals repeat in blocks of predictable lengths.

Table 2

n		Finite sum of the original number
1	.5	5
2	.25	7
3	.125	8
4	.0625	13 → 4
5	.03125	11 → 2
6	.015625	19 → 10 → 1
7	.0078125	23 → 5
8	.00390625	25 → 7
9	.001953125	26 → 8
10	.0009765625	40 → 4
11	.00048828125	38 → 11 → 2
12	.000244140625	28 → 10 → 1

For example, notice in Table 2 that digits in the third diagonal repeat in blocks of two; namely 1,6, while digits along the fourth diagonal repeat in blocks of four; namely 0,3,5,8. Furthermore, the digits in a block occur in complementary pairs.

Along the fourth diagonal,



0 and 5 are complements and 3 and 8 are complements.

Conjecture 3. The n^{th} diagonal repeats in blocks of 2^{n-2} (for $n \geq 2$) digits. Furthermore, the digits occur in complementary pairs, one in the upper half of the block and the other 2^{n-3} rows below in the lower half of the block.

Proof. The proof of conjecture 3 follows directly from the proof of conjecture 2. We showed that the decimal expansions of any two powers of 2 separated in Table 1 by 2^{k-2} rows agree in their last k places. Since we also know that 2^{-n} contains exactly n digits in its decimal expansion, each row contains one more digit than its predecessor. Therefore, the digits in the n^{th} diagonals also repeat in blocks of 2^{n-2} digits ($n \geq 2$).

In addition, by removing the factor of 10^k from D on line(2) in the proof of conjecture 2 (in order to determine the nature of the last nonzero digits in D), it is evident that

$$\frac{D}{10^k} = 5^{n-k} \cdot Q \equiv Q \bmod 5 .$$

Hence, the last nonzero digits retained (which will of course fall on the $(k+1)^{\text{th}}$ diagonal) are congruent mod 5 (that is, they are complements). They are separated by 2^{k-2} rows which is indeed one-half of the length of the $(k+1)^{\text{th}}$ diagonal block.

Finite Sums Pattern

Looking back at Table 2, we note that if the digits of a number are added, and if the digits of the sum are added, and so on, until a single digit number is obtained, that the finite sums of the digits of the original numbers follow a pattern.

Conjecture 4. The finite sums of the digits of successive expansions of 2^{-n}, starting with $n = 1$, repeat every sixth row in the sequence 5,7,8,4,2,1.

For example,

$$\begin{array}{r}
15625 \longrightarrow 19 \longrightarrow 10 \longrightarrow 1 \\
\underline{796} \longrightarrow 22 \longrightarrow 4 \\
12437500 \longrightarrow 22 \longrightarrow 4
\end{array}$$

Since we know that the decimal expansion of 2^{-n} has exactly the same digits as 5^n and $5^6 = 15625$, the finite sums must repeat every sixth row.

Proof. Assume that it is known that the product of a number whose finite sum is x and a number whose finite sum is one is a number whose finite sum is x. [The proof is left as an exercise for the reader.]

Summary

The patterns discussed in this article may or may not have practical applications. However, like so many numerical observations, they are compelling in and of themselves. Furthermore, they give students the opportunity to practice both inductive and deductive reasoning.

I have discussed only four conjectures. Perhaps you found more. One path not explored in this article is the possible existence of patterns along the columns.

Good luck with your explorations.

Acknowledgement

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The Transformation of an Inconsistent Linear System into a Consistent System

by

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Miguel Paredes received his PhD from the University of Utah. His interests include mathematical economics, modeling and education. Mohammad Fatehi received his PhD from New Mexico State University. His area of interest is operation research and statistics. Richard Hinthorn's area of interest is algebra.

A review of college textbooks in mathematics reveals that an impressive majority of the applied examples and exercises of linear systems of equations lead to consistent systems. Most teachers and some inquisitive students quickly notice that in certain cases, slight variations in the data provide inconsistent systems. With a few exceptions no further considerations are made once a system is found to be inconsistent. However, Leon (1986) does provide an example of a transformation of an inconsistent system into a consistent system. Example 1 is a real agricultural decision making problem with data taken from an Agricultural Experiment Service Report (1984) that yields an inconsistent system. Examples like this have motivated the authors to ask: What is the theoretical basis for transforming an inconsistent system into a consistent system?

Let us consider a nonhomogeneous inconsistent system with augmented matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdot\cdot & a_{2n} & b_2 \\ . & . & \cdot\cdot & . & . \\ a_{m1} & a_{m2} & \cdot\cdot & a_{mn} & b_m \end{bmatrix} \text{, with } m \geq n$$

and its echelon form

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdot\cdot & c_{1n} & d_1 \\ c_{21} & c_{22} & \cdot\cdot & c_{2n} & d_2 \\ . & . & \cdot\cdot & . & . \\ c_{r1} & c_{r2} & \cdot\cdot & c_{rn} & d_r \\ 0 & 0 & .. & 0 & d_{r+1} \\ . & . & \cdot\cdot & . & . \\ 0 & 0 & .. & 0 & d_m \end{bmatrix}.$$

We claim that the system with augmented matrix

$$A' = \begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdot\cdot & a_{2n} & b_2 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ a_{r1} & a_{r2} & \cdot\cdot & a_{rn} & b_r \\ a_{r+1\ 1} & a_{r+1\ 2} & \cdot\cdot & a_{r+1\ n} & b_{r+1}-d_{r+1} \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot\cdot & a_{mn} & b_m-d_m \end{bmatrix}.$$

is consistent.

In order to prove our claim let us observe that since interchanging the rows in the augmented matrix of a system produces the augmented matrix of an equivalent system, we may assume without loss of generality that **A** is arranged in such a way that the last m-r rows in **A** give rise to the last m-r rows in **C** with the preservation of the order through the transformation. We may also assume without loss of generality that the last m-r rows in **C** are obtained by subtracting multiples of the top r rows of **A**. If we perform on **A′** the same operations that were performed on **A** to get **C**, we will get

$$C' = \begin{bmatrix} c_{11} & c_{12} & \cdot\cdot & c_{1n} & d_1 \\ c_{21} & c_{22} & \cdot\cdot & c_{2n} & d_2 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ c_{r1} & c_{r2} & \cdot\cdot & c_{rn} & d_r \\ 0 & 0 & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ 0 & 0 & \cdot\cdot & 0 & 0 \end{bmatrix}.$$

Example 1. A farmer who is planning to cultivate three types of vegetables needs to find out how many acres of land he should allocate in order to use all the resources listed in the last column in the table below. The table also shows the amount of each resource required per acre for each vegetable. The amounts of nitrogen and phosphate are measured in pounds and labor in hours of labor-machinery.

	Honeydews	Yellow Onions	Lettuce	Total
Land (acre)	x	y	z	220
Nitrogen (lb/acre)	120	150	180	29100
Phosphate (lb/acre)	180	80	80	32600
Labor (hrs/acre)	4.97	4.45	4.65	480

The solution of the farmer's problem is the solution of the system

$$\begin{aligned} x + y + z &= 220 \\ 120x + 150y + 180z &= 29100 \\ 180x + 80y + 80z &= 32600 \\ 4.97x + 4.45y + 4.65z &= 480\,. \end{aligned}$$

An echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 1 & 1 & 220 \\ 0 & 1 & 2 & 90 \\ 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & -581 \end{bmatrix} .$$

Thus, the original system is inconsistent, and the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 220 \\ 120x & + & 150y & + & 180z & = & 29100 \\ 180x & + & 80y & + & 80z & = & 32600 \\ 4.97x & + & 4.45y & + & 4.65z & = & 1061 \end{array}$$

is consistent. This means that if total labor-machinery hours is 480 during the planting season, then the farmer's problem has no solution, but if the number of labor-machinery hours is redefined as 1061, then the farmer's problem has a unique solution $x = 150$, $y = 50$, and $z = 20$; i.e., the number of acres allocated for honeydews, yellow onions, and lettuce respectively are 150, 50, and 20. It is interesting to observe that this solution is independent of the initial total amount of labor hours. This may be seen by replacing 480 by a parameter α in the original system. The resulting entry in the fourth row and fourth column of the echelon form of the new augmented matrix is $\alpha - 1061$. Therefore, the adjusted total labor is $\alpha - (\alpha - 1061) = 1061$ hours.

Suppose we fix the level of land, phosphate, and labor as stated in Example 1, and we wish to find the amount of adjustment for nitrogen so that the farmer's problem has a solution. This will be presented as Example 2.

Example 2. Let us consider the following system of equations:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 220 \\ 180x & + & 80y & + & 80z & = & 32600 \\ 4.97x & + & 4.45y & + & 4.65z & = & 480 \\ 120x & + & 150y & + & 180z & = & 29100 \end{array} .$$

Notice that we have reorganized the equations. The reduced echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 0 & 150 \\ 0 & 1 & 0 & 2955 \\ 0 & 0 & 1 & -2885 \\ 0 & 0 & 0 & 87150 \end{bmatrix} .$$

It provides no solution to the farmer's problem, because the solution involves a negative number of lettuce and the new total amount of nitrogen in the transformed consistent system is also negative.

In consulting work, farmers are interested in finding out whether some of the total resources could be re-allocated so that the system becomes consistent. As an illustration we shall consider the following augmented matrix, where we keep the available land fixed at 220, and represent the quantities of phosphate and labor-machinery by the parameters α and β respectively. The augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 220 \\ 180 & 80 & 80 & \alpha \\ 4.97 & 4.45 & 4.65 & \beta \\ 120 & 150 & 180 & 29100 \end{bmatrix}$$

has a reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & \dfrac{-17600+\alpha}{100} \\ 0 & 1 & 0 & \dfrac{2416700+8\alpha-2500\beta}{500} \\ 0 & 0 & 1 & \dfrac{-2218700-13\alpha+2500\beta}{500} \\ 0 & 0 & 0 & \dfrac{12394200+108\alpha-15000\beta}{100} \end{bmatrix}.$$

The total amount of nitrogen required for the transformed consistent system to have a solution is $\dfrac{-9484200-108\alpha+15000\beta}{100}$.

This number must be positive, and the first three entries of the last column of the reduced echelon form must be positive or zero. Observe also that these first three entries must be less than or equal to the total available land. Hence α and β must satisfy the following system of inequalities:

$$0 \le \frac{-17600+\alpha}{100} \le 220$$

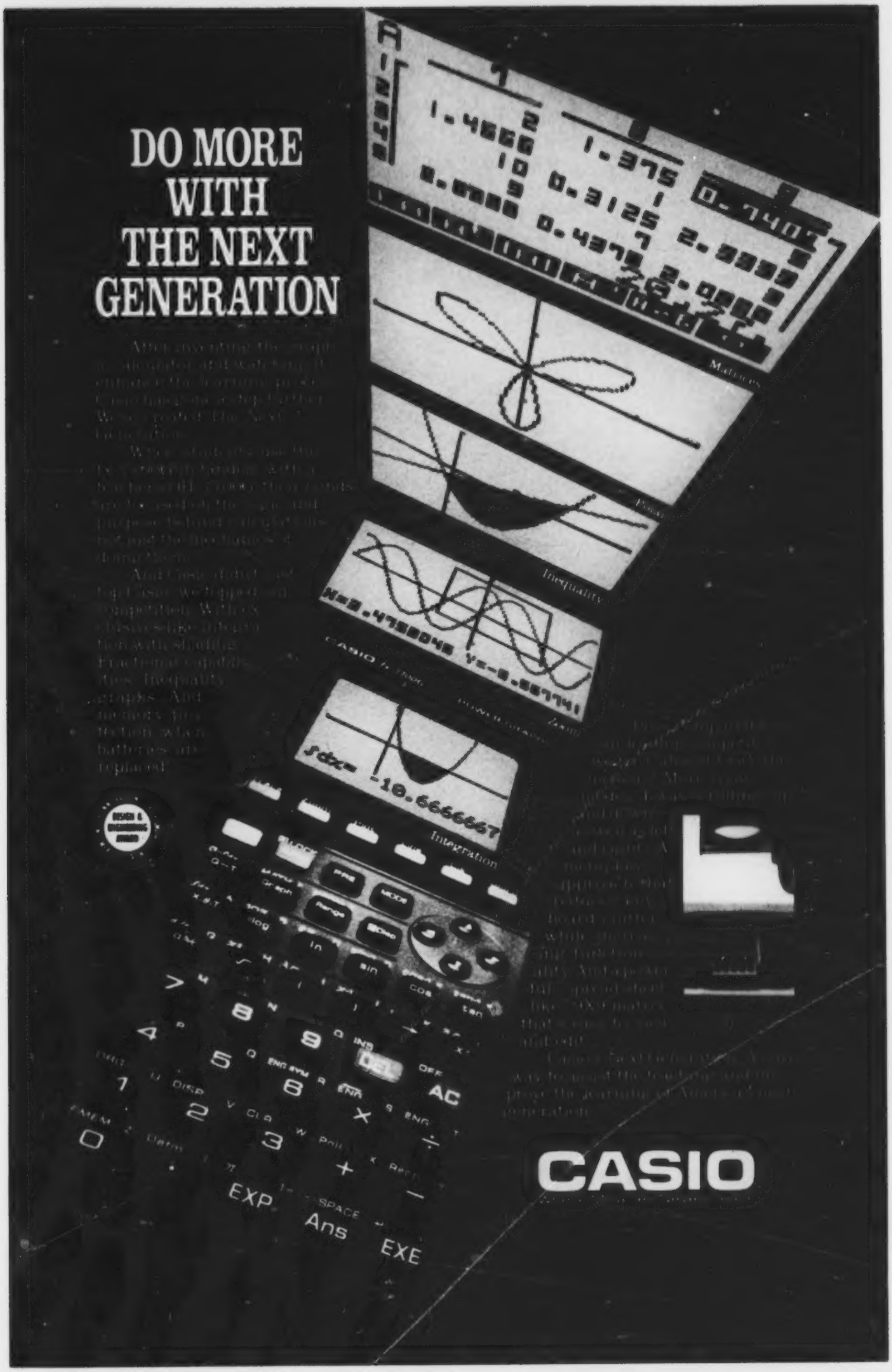
$$0 \le \frac{2416700+8\alpha-2500\beta}{500} \le 220$$

$$0 \le \frac{-2218700-13\alpha+2500\beta}{500} \le 220$$

$$-9484200 - 108\alpha + 15000\beta > 0$$

The solution of this system of inequalities in the α, β plane consist of the boundaries as well as the interior of the triangle with vertices (17600, 979), (17600, 1023), and (39600, 1093.4). For example, if $\alpha = 18000$ and $\beta = 1000$, the total amount of nitrogen in the consistent system is 29100 + 6618 = 35718; the numbers of acres allocated for honeydews, yellow onions, and lettuce respectively are 4, 121.4, and 94.6.

The material included here can be used to produce interesting exercises at all levels where matrices are used to solve linear systems of equations from high school algebra to college linear algebra. At the more elementary level the theory can be presented for linear systems with two or three equations and two variables, depending upon the students. At a more advanced level we can ask students to apply the theory in this paper to parametrize all the constants for examples with three variables and three or



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more equations. In linear algebra classes, the connection explored in Example 2 can be exploited by producing a similar system of linear inequalities when the system of equations has a larger size.

It can be observed from Example 2 that a problem involving a system of linear equations can be approached using a system of inequalities. This system of inequalities may be viewed as a set of constraints. In this case the system of the inequalities is a closed convex set in the plane. The problem of finding a best solution arises; hence we have an optimization problem, which can be solved by using linear programming methods. We shall not develop this any further here, because it is the subject matter of a future paper.

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Deriving the Area of a Circle

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Historically, one of the oldest, difficult problems in mathematics was finding the area of a circle. (Before reading further, reflect on how you would derive the formula.) Today the problem is often solved in integral calculus. Specifically, using symmetry and the equation of a circle with center at the origin and radius $r > 0$, the area, A, is given by the integral

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx\,. \tag{1}$$

Substituting $x = r\sin\theta$, the integral in (1) simplifies to

$$A = 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta\,. \tag{2}$$

Using the trigonometric identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}\,, \tag{3}$$

the integral is easily evaluated to give the well known formula,

$$A = \pi r^2. \tag{4}$$

While this approach appears to derive in a simple, concise way the required formula, the derivation is, in fact, invalid! Specifically, the transformation from (1) to (2) and the integration of (3) use the result

$$\frac{d\,(\sin\theta)}{d\,\theta} = \cos\theta\,. \tag{5}$$

While the derivation of (5) appears in most calculus texts, part of the derivation is needed to show why the integral derivation of the area of a circle is invalid. The derivation of (5) begins, of course, with the definition of the derivative and, after some trigonometric substitutions, gives the limit

$$\lim_{h\to 0}\frac{\sin h}{h}\,. \tag{6}$$

Figure 1 is used to find the limit in (6).

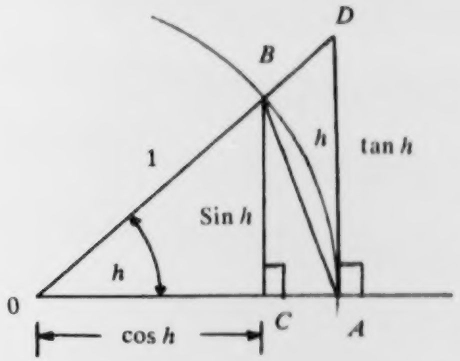

Figure 1. Derivation of the limit of (sin *h*)/*h* as *h* goes to zero

From Figure 1,

$$\text{area of } \Delta OAB < \text{area of sector } OAB < \text{area of } \Delta OAD. \qquad (7)$$

The area of sector *OAB* equals

$$\frac{h}{\text{Circumference of circle}} \cdot \text{Area of circle} = \frac{h}{2\pi \cdot 1} \cdot \pi \cdot 1^2 = \frac{h}{2}. \qquad (8)$$

See a calculus text for the remainder of the proof that the limit is one (e.g., Finney & Thomas, 1990, p. 91). As the derivation of the area of a sector uses the area of a circle (as well as the circumference of a circle), the derivation of the area of a circle using the integral in (1) is circular and hence, invalid!

Derivation of the Area of a Circle

There are two commonly used methods for finding the area of a figure when an area formula is not known. One is to decompose the figure into disjoint subfigures whose areas can be added. For example, from the area of triangles, the area of a regular *n* sided polygon can be obtained. A second way to find the area is to inscribe and circumscribe the figure successively with shapes whose areas converge to the same number. This bounds the figure's area between two sequences which converge to the same value, the area of the figure. For a circle, we use the second procedure and inscribe and circumscribe regular polygons. Figure 2 shows isosceles triangles forming the *n* sided inscribed and circumscribed regular polygons for a circle with radius *r*.

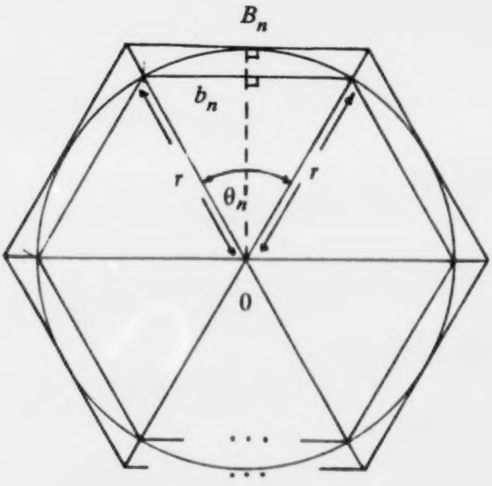

Figure 2. Inscribed and circumscribed regular polygons for a circle of radius *r*

The areas of the two polygons bound the area of the circle. For the inscribed polygon, the Pythagorean theorem gives the altitude of an isosceles inscribed triangle, so the area of one isosceles inscribed triangle is

$$\frac{b_n}{2}\sqrt{r^2-\left(\frac{b_n}{2}\right)^2}, \tag{9}$$

where b_n is the length of the base of the inscribed triangle. Multiplying by the number of triangles (all congruent) in the regular n sided polygon, n, gives the area of the inscribed polygon, a_n. Since n times the length of the base of the triangle, b_n, gives the perimeter of the inscribed polygon, p_n, the area of the inscribed polygon is

$$a_n = \frac{p_n}{2}\sqrt{r^2-\left(\frac{b_n}{2}\right)^2}\,. \tag{10}$$

For the circumscribed polygon, each of its n triangles in Figure 2 has area

$$\frac{1}{2}B_n r, \tag{11}$$

where B_n is the length of the base and r is the altitude. We want to express B_n in terms of b_n. Using the Pythagorean theorem and the fact that the triangles in the inscribed and circumscribed polygons are similar,

$$\frac{B_n}{r} = \frac{b_n}{\sqrt{r_2-\left(\frac{b_n}{2}\right)^2}}\,. \tag{12}$$

Multiplying (11) by n and using (12), the area of the circumscribed polygon, A_n, is

$$A_n = \frac{nr^2 b_n}{2\sqrt{r^2 - \left(\frac{b_n}{2}\right)^2}} = \frac{p_n r^2}{2\sqrt{r^2 - \left(\frac{b_n}{2}\right)^2}} . \tag{13}$$

The area of the circle, A, is bounded below by a_n in (10) and above by A_n in (13). To find the limits as n goes to infinity of a_n and A_n, we consider subsequences of each of the sequences. We consider the subsequences a_{2^k} and A_{2^k}, areas of inscribed and circumscribed regular polygons with $2^2, 2^3, \cdots, 2^k, \cdots$ sides. For these subsequences, the first polygon is a square, and subsequent polygons are obtained by doubling the number of sides of the previous polygon.

First a proof is given that the subsequences converge. Then a proof is given that the sequences themselves converge.

Figure 3 shows triangles for two successive inscribed polygons with 2^k and 2^{k+1} sides. AB has length b_{2^k}. AD and BD have length $b_{2^{k+1}}$. By the triangle inequality, $b_{2^{k+1}} + b_{2^{k+1}} > b_{2^k}$, make an increasing sequence, bounded above by the perimeter of any circumscribed polygon. Thus the limit of p_{2^k} exists. Let p_∞ denote the limit of the inner perimeters, $2^k b_{2^k}$. It will be shown that p_∞ is the circumference of the circle. However, first it is necessary to show that the limit of b_{2^k}, as k goes to infinity, is zero.

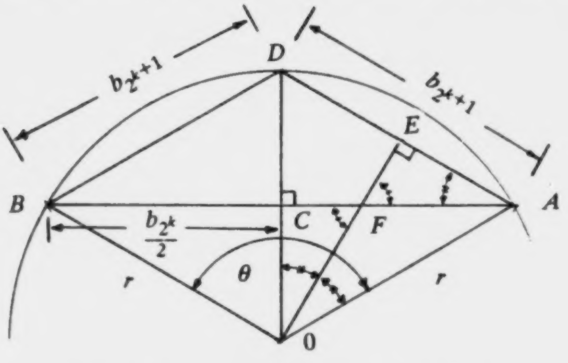

Figure 3. Inscribed polygons with 2^k and 2^{k+1} sides

In Figure 3, let $m(\angle BOA) = \theta$. Then $m(\angle DOA) = \theta/2$, and $m(\angle DOE) = \theta/4$. Angles $\angle OCA$ and $\angle AEO$ are right angles with $\angle AFE$ and $\angle CFO$ congruent. Triangles AEF and OCF are similar, giving that $\angle DAC$ and $\angle DOE$ have measure $\theta/4$. Thus,

$$\frac{b_{2^k}}{2b_{2^{k+1}}} = \cos(\angle DAC) = \cos\frac{\theta}{4} . \tag{14}$$

For the subsequence a_{2^k}, the first inscribed polygon is a square. Thus,

$$\frac{\theta}{4} \le \frac{90°}{4} < 45° . \tag{15}$$

Since $\cos\theta$ decreases in $[0°, 90°]$,

$$\cos\frac{\theta}{4} \ge \cos\frac{90°}{4} > \cos 45° = \frac{\sqrt{2}}{2} . \tag{16}$$

(Radians can not be used since radian measure assumes the perimeter of a circle is 2π.) Combining (14) and (16),

$$b_{2^k}\frac{\sqrt{2}}{2} > b_{2^{k+1}} . \tag{17}$$

Hence, the positive subsequence b_{2^k} decreases and goes to zero as k goes to infinity.

Since the limit of b_{2^k} is zero when k goes to infinity, the limits of a_{2^k} and A_{2^k} as k goes to infinity are (from (10) and (13) respectively)

$$\frac{p_\infty r}{2} \quad \text{and} \quad \frac{p_\infty r^2}{2r} = \frac{p_\infty r}{2} , \tag{18}$$

respectively. The upper and lower bounds of the area of the circle are equal so the area of the circle is this common value.

To show p_∞ is the circumference, C, consider the perimeters of the inscribed and circumscribed polygons, $2^k b_{2^k}$ and $2^k B_{2^k}$, respectively. Using (12), the ratio $\frac{2^k B_{2^k}}{2^k b_{2^k}}$ has a limit equal to one as k goes to infinity. Since the circumference of the circle, C, lies between the perimeters of the inscribed and circumscribed polygons and both sequences of perimeters have the same limit p_∞, the limit p_∞ equals C. This result with (18) gives the following:

Theorem 1. For a circle with radius $r > 0$, A its area, and C its circumference,

$$A = \frac{C \cdot r}{2} . \tag{19}$$

To complete the derivation, we use the functional notation $A(r)$ and $C(r)$ rather than the simpler A and C for the area and circumference of a circle, respectively.

Theorem 2. For a circle with radius $r > 0$, $A(r)$ its area, and $C(r)$ its circumference,

$$A(r) = r^2 \cdot A(1) . \tag{20}$$

Proof. In Figure 2, triangles in the inscribed (or circumscribed) polygons when the radius of the circle is r are similar to the corresponding triangles when the radius is one. Thus, the ratio of the area of corresponding triangles in the inscribed (or circumscribed) polygons with circle radius r and $r = 1$ and in turn the ratio of the area of the inscribed (or circumscribed) polygons equal r^2. Therefore,

$$a_{2^k}(r) = r^2 a_{2^k}(1) \text{ and } A_{2^k}(r) = r^2 A_{2^k}(1)\,. \tag{21}$$

Letting k go to infinity gives

$$A(r) = r^2 A(1) \text{ and } \frac{A(r)}{r^2} = \frac{A(1)}{1}\,. \tag{22}$$

$A(r)/r^2$ and $A(1)/1$ equal a unitless constant. Today we call this constant π. Theorem 1 and (22) give that the circumference of a circle, $C(r)$, satisfies

$$\frac{C(r)}{2r} = \frac{A(r)}{r^2} = \pi\,. \tag{23}$$

Equation (23) gives the desired theorem. (We drop the functional notation here.)

Theorem 3. For a circle with radius $r > 0$, A its area, and C its circumference,

$$A = \pi r^2 \qquad \text{and} \qquad C = 2\pi r. \tag{24}$$

Now a proof will be given that the original sequences a_n and A_n converge. A proof is required since a subsequence can converge, but the original sequence diverge. To show a_n converges, inscribe a circle in the n sided polygon in Figure 2. The radius of such a circle equals the altitude of a triangle in the inscribed polygon, so its area is

$$\pi\left(\sqrt{r^2 - \left(\frac{b_n}{2}\right)^2}\,\right)^2 = \pi\left(r^2 - \left(\frac{b_n}{2}\right)^2\right) \tag{25}$$

As b_n is a decreasing, positive sequence, it converges to zero since the subsequence b_{2^k} converges to zero. The area of this inscribed circle and the area of the circle of radius r, πr^2, bound the area of the inscribed polygon, a_n. As the limit of the areas of the circles inscribed in the inscribed polygons goes to πr^2, the sequence a_n has limit πr^2. A similar proof shows the areas of the circumscribed polygons, A_n, also has this limit.

A final question concerns the use of polygons. We could inscribe and circumscribe sequences of other geometric figures that exhausts the area of the circle. Can we show that the sequences of areas of these figures would have the same limit πr^2?

Theorem 4. Let I_n be a sequence of areas of figures each of which is interior to a circle of radius r and E_n be a sequence of areas of figures each of which contains the circle. If the two sequences have the same limit, then the limit is πr^2.

Proof. The area of the circle lies between I_n and E_n, so

$$I_n \leq \pi r^2 \leq E_n . \qquad (26)$$

Since the two sequences have the same limit, the limit of each must equal πr^2.

Theorem 4 shows that the area of the circle, of course, does not depend on the fact that polygons were used in the derivation of the formula.

Historical Notes on the Area of a Circle

Euclid considers the area of circles in the *Elements*. He defines a circle in Book I (Definition 15) and shows that a square can be inscribed and circumscribed about a circle in Book IV. He obtains a result similar to Theorem 2 in Book XII (Proposition 2), showing that the ratio of the areas of two circles equals the ratio of the squares of their diameters. (As the diameter is twice the radius, the ratio also equals the ratio of the squares of the two radii.) This result seems to have been obtained first by Hippocrates of Chios about 450 B.C., about 150 years prior to the *Elements*. See Heath (1926, Vol III). At the time of Euclid, the Greeks also knew that the ratio of the circumferences of circles equals the ratio of their diameters. Thus by the time of Euclid, the Greeks knew that there were constants α and β so

$$A = \alpha r^2 \qquad \text{and} \qquad C = 2\beta r . \qquad (27)$$

What Euclid did not find was any relationship between the two constants! It was not until about 225 B.C., about 75 years after the *Elements*, that Archimedes in his famous work *Measurement of A Circle* proved Theorem 1. See Heath (1950). This established that the two constants were equal! (Today we denote the common value, of course, π.) Archimedes – like Euclid in Book XII – used Eudoxus' "method of exhaustion." This is the procedure used in this paper. (The inscribed figures "exhaust" the area of the figure.) Archimedes assumed that the inscribed polygons approximate the area of the circle as closely as desired. In this paper, this result is established using limits. Dunham (1990, Ch. 4) gives a more detailed discussion of the Greek knowledge of the area of the circle at the time of Euclid and a discussion of Archimedes' proof of Theorem 1.

Conclusion

Deriving the area of a circle is an interesting exercise in geometry. If it is derived in calculus, it should be considered in the first part of the course and not derived using the integral in integral calculus. The integral considered at the beginning of this article is appropriate for deriving the area of ellipses but not of circles. The historical derivation of the formula is also interesting. It shows the difficulty of obtaining the formula that today is so well known.

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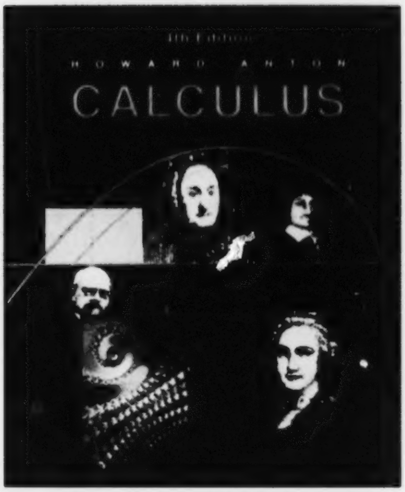

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The Hollywood Bowl uses a "stacked parking" format in which each car is parked directly behind the other with all cars facing the same direction. Moreover, the cars are situated closely side-by-side so that the only way to exit is to wait until all of the cars ahead of yours have left. We have attended excellent concerts there, but have always been struck by how long it seems to take to get out of the parking lot. Although many of the drivers ahead of us are usually ready to go, by the time the **last** such driver arrives, it seems that most of those behind us are also there waiting to leave. Reflecting upon this queuing phenomenon, we formulated the two questions below, assuming throughout that we are the first to return to our car.

Question 1. Suppose n cars are parked in a line and ours is in position k from the front. We cannot drive out until the $k-1$ drivers in front of us have arrived. Assuming these drivers arrive in a random order, what is the expected number of drivers that need to arrive until we are clear to leave?

Problems similar to this appear in the literature. For example, in Meyer (1970) is found the question (page 154, problem #7.8), "A lot is known to contain 2 defective items and 8 nondefective items. If these items are inspected at random, one after the other, what is the expected number of items that must be chosen *for inspection* in order to remove all the defective ones?" Note, however, that this question is slightly different mathematically since at most 7 items would have to be inspected. In Ross (1984) is found a similar question (page 309, problem #3). It is not always apparent that rather unexciting textbook problems can occur in forms that are considerably more meaningful and interesting.

Solution. For $j = 0, 1, 2, \ldots, n-k$, the probability we have to wait for exactly $j+k-1$ drivers is identical to the probability that the $(j+k-1)$-th driver to arrive is one of the $k-1$ drivers parked in front of us and among the first $j+k-2$ drivers to arrive are the $k-2$ other drivers parked in front of us. This probability is given by

$$\frac{(k-1)\binom{k-2}{k-2}\binom{n-k}{j}}{(n-1)\binom{n-2}{j+k-2}}$$

For example, if 12 cars are parked in a line and ours is in position 5 from the front, the probability we have to wait for exactly 8 drivers $(n = 12, k = 5, \text{and } j = 4)$ is

$$\frac{4\binom{3}{3}\binom{7}{4}}{11\binom{10}{7}} .$$

Therefore, the expected number of drivers for whom we must wait is

$$E=\sum_{j=0}^{n-k}(j+k-1)\frac{(k-1)\binom{n-k}{j}}{(n-1)\binom{n-2}{j+k-2}}=\frac{k-1}{n-1}\sum_{j=0}^{n-k}((j+k-1)\frac{\frac{(n-k)!}{j!(n-k-j)!}}{\frac{(n-2)!}{(j+k-2)!(n-k-j)!}}$$

$$=\frac{(k-1)(n-k)!}{(n-1)!}\sum_{j=0}^{n-k}\frac{(j+k-1)!}{j!}=\frac{(k-1)(n-k)!(k-1)!}{(n-1)!}\sum_{j=0}^{n-k}\frac{(j+k-1)!}{j!(k-1)!}$$

$$=\frac{(k-1)(n-k)!(k-1)!}{(n-1)!}\sum_{j=0}^{n-k}\binom{k-1+j}{j}.$$

But,

$$\sum_{j=0}^{n-k}\binom{k-1+j}{j}=\binom{n}{n-k} .$$

This last identity is a consequence of the fundamental recurrence property,

$$\binom{m-1}{r-1}+\binom{m-1}{r}=\binom{m}{r} ,$$

of Pascal's triangle. Therefore,

$$E=\frac{(k-1)(n-k)!(k-1)!}{(n-1)!}\,\frac{n!}{k!(n-k)!}=\frac{n(k-1)}{k}=n-\frac{n}{k} .$$

For example, if 30 cars are parked in the line and ours is in position 10, we can expect to wait for 27 of the 29 other drivers until we are clear to leave. Actually, such a long wait should not be too surprising since the probability is 9/29 (almost 1/3) that the last driver to arrive will be parked in front of us. Our feeling that it takes so long to get out is not merely impatience.

Question 2. Suppose n cars are parked in a line and ours is in one of these n equally likely positions. We cannot drive out until the drivers in front of us have arrived. Assuming these drivers arrive in a random order, what is the expected number of drivers that need to arrive until we are clear to leave?

Solution. We found above that if we are in position k, the expected number of drivers that need to arrive until we are clear to leave is $n - n/k$. The average of these expected waits is

$$\frac{1}{n}\sum_{k=1}^{n}\left(n-\frac{n}{k}\right)=\sum_{k=1}^{n}\left(1-\frac{1}{k}\right)=n-\sum_{k=1}^{n}\frac{1}{k}\approx n-1-\ln\frac{n+0.5}{1.5}$$

(see Liss, McMillan, & Milton (1990)). Some specific examples are given in Table 1.

Table 1

n	$n-\sum_{k=1}^{n}\frac{1}{k}$	$n-1-\ln\frac{n+0.5}{1.5}$
10	7.07	7.05
20	16.40	16.39
30	26.00	25.99
100	94.81	94.80
200	194.12	194.10

Conclusion

We assumed in the beginning that we are the first to return to our car. We now see that, since we most likely will have to wait for almost all of the drivers to arrive before we can leave, it rarely does any good to leave the concert early to get a head start.

That the answer to Question 2 involves the number e, the base of the natural logarithms, reminds us of how often this important number arises in probability. For several other examples, the reader is referred to Shultz & Leonard (1989).

Please send any correspondence regarding this article to Harris Shultz at California State University at Fullerton, Fullerton, CA 92634.

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An Inductive Proof of Fermat's Little Theorem

by Jeff Suzuki
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Fermat's little theorem states that given any prime number p and any integer a not a multiple of p, p divides $a^{p-1}-1$. This theorem is often used to test for primeness of large numbers, as it is easier to determine the congruence class of a large number than to divide it by every possible prime factor. (Unfortunately, there also exist numbers for which the theorem holds but the number p is not prime; see the note after Lemma 2.) A more general theorem is that for any prime number p, p divides $a^{p}-a$ for any value of a. Fermat's little theorem as well as the more general theorem can be proven using an argument based on congruence classes; however, with the help of three nearly self-evident lemmas, a simpler and more intuitive proof is presented below. We first prove the more general theorem; the main theorem then becomes a corollary.

Lemma 1. If n divides ab, and $(n,a) = 1$ (i.e., the greatest common divisor of n and a is one), the n divides b.

Proof. $(n,a) = 1$ implies that 1 can be expressed as a linear combination of n and a. Thus, for example, $xn + ya = 1$. Multiplying both sides by b we get $xnb + yab = b$; since n divides xnb and n divides yab (by assumption), it must also divide b.

Lemma 2. For p prime and $1 \le j \le p-1$, p divides $\binom{p}{j}$.

Proof. For p prime, we have $\binom{p}{j} = \dfrac{p!}{j!(p-j)!} = \dfrac{p(p-1)!}{j!(p-j)!}$. Obviously, $(p, j!\,(p-j)!) = 1$. Thus, by lemma 1, $j!\,(p-j)!$ divides $(p-1)!$, say $\dfrac{(p-1)!}{j!(p-j)!} = k$. Thus $\binom{p}{j} = pk$, and thus p divides $\binom{p}{j}$.

Note that there exist numbers k and j, where k is not prime, such that k divides $\binom{k}{j}$. For example, 12 divides $\binom{12}{6}$. The existence of nonprimes for which lemma 2 is true accounts for the nonprimes for which Fermat's little theorem holds.

Lemma 3. If a divides b, and c divides b, and $(a,c) = 1$, then ac divides b.

Proof. Our first statement implies that $b = na$; our second implies that $b = mc$. Also we know that $(a,c) = 1$ implies $ax + cy = 1$ for some x,y. Multiplying through by b, we get $abx + cby = b$; substituting we have $amcx + cnay = b$; clearly ac divides the left, so it divides the right also.

Theorem. If p is prime, then p divides $n^{p}-n$.

Proof. We use induction: p divides $1^p - 1$. Suppose p divides $n^p - n$. Then we have

$$(n+1)^p - (n+1) = n^p + \binom{p}{1} n^{p-1} + \binom{p}{2} n^{p-2} + \ldots + \binom{p}{p-2} n^2 + pn + 1 - n - 1$$

by the binomial expansion of $(n+1)^p$. This simplifies to:

$$(n+1)^p - (n+1) = n^{p-n} + \binom{p}{1} n^{p-1} + \binom{p}{2} n^{p-2} + \ldots + \binom{p}{p-2} n^2 + pn\,.$$

By the assumption, p divides $n^p - n$. By lemma 2, p divides all of the $\binom{p}{j}$ terms. Since p divides pn, p divides the left side. Therefore, by induction, p divides $n^p - n$.

Fermat's Little Theorem. If p is prime and n is not a multiple of p, then p divides $n^{p-1} - 1$.

Proof. p divides $n^p - n$, and, obviously, so does n. Therefore (by lemma 3) pn divides $n^p - n$, and clearly then p divides $n^{p-1} - 1$.

A Swiftly Convergent Sequence to Pi

by Michael Schapira
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In order to approximate the value of π we consider the following sequences: the perimeters of regular polygons inscribed within and the perimeters of regular polygons circumscribed about a circle with radius one. As the number of sides, n, increases, each sequence converges monotonically to the value of the circumference of the unit circle, 2π. The sequence formed by taking one-half of the average of the two sequences converges to π. However, the rate of convergence is relatively slow; it requires $n = 1536$ sides to approximate π to five decimal places (see column 2 of Table 1). One may ask, can the rate of convergence be improved by using a weighted average, and if so, which weighted average would be best?

In addressing this question, we consider the behavior of the half-perimeters of the inscribed and circumscribed regular polygons as the number of sides increases. The half-perimeters are chosen because they converge to π rather than 2π.

Given a unit circle,

(a) let I_n denote the half-perimeter of an n-sided inscribed regular polygon; then

$$I_n = \tfrac{1}{2} n \left(2 \sin \frac{360^\circ}{2n}\right), \qquad \text{(See Figure 1)}$$

(b) and let C_n denote the half-perimeter of an n-sided circumscribed regular polygon; then

$$C_n = \tfrac{1}{2} n \left(2 \tan \frac{360^\circ}{2n}\right), \qquad \text{(See Figure 2)}$$

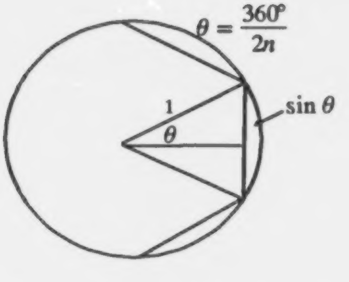


Figure 1. Regular inscribed polygon

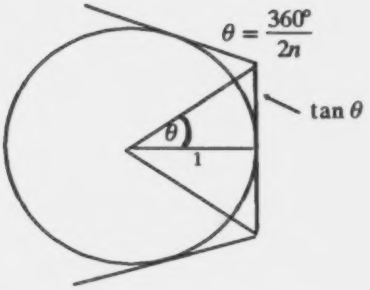


Figure 2. Regular circumscribed polygon

In Table 2 the values of I_n and C_n are presented for increasing values of n, together with their respective first differences, ΔI and ΔC. We can see that as n increases, I_n increases and C_n decreases and, furthermore, that the first differences, ΔI and ΔC, approach zero. When summing the elements of two series if, from some point on, the first differences are equal and opposite in sign to some desired accuracy, then convergence of the sum is achieved to that accuracy. In Table 2, we see that in no row are the first differences equal and opposite in sign. However, we note that from an early point on the absolute value of ΔC is close to two times the corresponding ΔI. This observation is made explicit in the sixth column where the ratio $\left|\frac{\Delta C}{\Delta I}\right|$ is presented and where it can be seen to approach the value of two as n increases. This means that if we give a weight to I_n which is twice the weight we give to C_n, then the weighted first differences will cancel. In other words, the weighted average $(2I_n + C_n)/3$ converges to π more rapidly than $(I_n + C_n)/2$ (see Table 1).

Table 1: The Average and the Weighted Average

Sides n	$(I_n + C_n)/2$	$(2I_n + C_n)/3$
3	3.89711431702	3.46410161513
6	3.23205080757	3.15470053838
12	3.16060942520	3.14234913055
24	3.14614427768	3.14163905622
48	3.14271820908	3.14159554040
96	3.14187327528	3.14159283382
192	3.14166276114	3.14159266485
384	3.14161017749	3.14159265430
768	3.14159703437	3.14159265363
1536	3.14159374877	3.14159265359

The value of π to eleven decimal places is 3.14159265359.

Table 2: The Half-Perimeters, First Differences and Ratios

	Inscribed Polygon		Circumscribed Polygon		
Sides	Half-Per.	1st Diff.	Half-Per.	1st Diff.	
n	I_n	ΔI	C_n	ΔC	$\left\lvert\frac{\Delta C}{\Delta I}\right\rvert$
3	2.598076		5.196152		
6	3.	0.401924	3.464102	-1.732051	4.3094
12	3.105829	.105829	3.215390	-.248711	2.3501
24	3.132629	.026800	3.159660	-.055730	2.0795
48	3.139350	.006722	3.146086	-.013574	2.0194
96	3.141032	.001682	3.142715	-.003372	2.0048
192	3.141452	.000421	3.141873	-.000842	2.0012
384	3.141558	.000105	3.141663	-.000210	2.0003
768	3.141584	.000026	3.141610	-.000053	2.0001
1536	3.141590	.000007	3.141597	-.000013	2.0000

Editor's Note: Mathematics teachers must be very creative in order to get some students interested in the subject matter. In the following "articles," Professors Wiranowski and Francis present some poetry that they suggest will help to spark the needed interest.

Poetry in Curve Sketching

by Jan Z. Wiranowski
Monroe Community College, Rochester, NY 14623

Half-way through the semester, when both the limits of endurance and those of calculus have been sufficiently tested, the time seems appropriate for a momentary change of pace. Curve-sketching techniques are being studied, and I use that opportunity to introduce my students to a different teach-and-study tool, a poem.

Explanatory remarks precede the presentation. Then, the poem, entitled "The Calc-U-Slalom Course," is read with a considerable amount of expression in order to generate student interest.

During the discussions which accompany this activity, students are frequently referred to the poem's notes. This reinforces the learning process and reassures the students that the amount of detail to be retained is quite manageable. It also renders the instructor's role more personal and more creative. Occasionally, the use of the poem is allowed during the exam.

The notes are generated by the poem's specially chosen vocabulary (*dome* → domain, *rang* → range, etc.). They provide, in a visually discernible manner, a concise and fairly complete summary of the curve-sketching techniques. The flow of the poem envisions movement along the turns of an imaginary slalom course. The sound of bells echoing in the valley completes the picture (ah, how they reverberate those younger days of teaching skiing in the Alps and Vermont!). The poem culminates with a transformation of the ski track into the anticipated curve.

The *dome*[1]
rang[2]
zeros,[3] so
y-intercept[4]
the runaway *asymptotes,*[5]
jump the chasm of
Zeno's Paradox

when
one is about to discover
the most *critical*[6] of the
impartial *numbers*[6];
in a breath-taking course
of *climb*[7] and *descent,*[8]
along the dreamlike *extrema,*[9]
with *concavities*[10] rising to the
very peaks, or sinking –
way down to the minima;
by way of an ominous
cornice (no time to pause and
marvel), beyond the cornice
looms a precipice; this course
must continue – onward; through
each and every *gate of inflection,*[11]
while in essence,
cautiously,
with quasi ski-edge perfection,
pencil in hand,
one begins betracing
the curve of suspense:

over strategically *plotted*
sitzmarks.[12]

NOTES:

1. Domain
2. Range, if possible
3. $f(x) = 0$: x-intercepts (roots), if possible
4. $f(0) = y$-intercept
5. Asymptotes:
   H.A.: $\lim_{x \to \pm\infty} f(x) = L$; $y = L$
   V.A.: $\lim_{x \to d\pm} f(d) = \pm\infty$; $x = d$ (from left or right)
6. Critical number(s): "c",
   if $f'(c) = 0$ or $f'(c)$ "undefined"
7. $f' > 0 \to f$ is increasing on interval
8. $f' < 0 \to f$ is decreasing on interval
9. Extrema:

First Derivative Test

$f(c)$: MAX
f: ↗ ↘
x: c
f': $+ \to -$

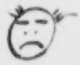
$f(c)$: MIN
f: ↘ ↗
x: c
f': $- \to +$

If the sign of f' remains the same on both sides of the critical number c, then $f(c)$ is neither MAX nor MIN.

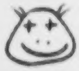
$$\left.\begin{array}{l}\text{Faces}\\\text{Below}\end{array}\right\}\begin{array}{l}f'': \text{pos./neg.} \to +/- \text{ for eyes}\\\text{mouth} \to \text{shape of the curve}\end{array}$$

Second Derivative Test

$f''(c) < 0$
MAX



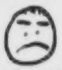
$f''(c) > 0$
MIN



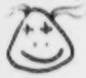
$f''(c) = 0 \to$ no conclusion

10. Concavities on an interval:

$f'' < 0$
Conc. Down



$f'' > 0$
Conc. Up



11. Points of inflection.
    If $f''(a)=0$ or $f''(a)$ "undefined," and concavity changes from down to up or vice versa in the adjacent intervals determined by the number a, then $(a, f(a))$ is a point of inflection.
12. Plot a few points, if needed.

Glossary (suggested meaning and/or imagery)

Dome: bell tower; **rang zeros:** the sound of bells echoing in the valley; **y-intercept the runaway asymptotes:** why chase a distant dream; **chasm of Zeno's paradox:** glacier's crevasse; **concavities:** clouds; **cornice:** a cusp on the curve; **gate of inflection:** a slalom gate; **sitzmark:** posterior's imprint in the snow (skiers' parlance).

Poetry in Mathematics

by Richard L. Francis
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Apart from its frequent entertainment value and occasional humorous phrases, there is much that emerges of an instructionally worthwhile nature in the use of verse in the mathematics classroom. We can oftentimes spark an interest in some who would otherwise prove indifferent. Students may participate not only by reading, but also by writing. Appropriate lines may, on casual reading, conceal subtle meaning so as to require a cautious but figurative interpretation. Furthermore, *key or clever word choices often remain imbedded in memory long after course details become fuzzy or forgotten.*

Consider the mathematical, historical, and poetic account of the primes which follows:

Primality

Primes in Ancient Times

'Twas a scene of glowing ambition,
This long-ago mathematician,
Who studied the primes,
Amidst ancient times,
With every divisor condition.

Eratosthenes,
the Mathematician

In fancy, the ponderer now sees,
A talented Er-a-tos-the-nes,
With rightness who found,
The distance around,
All the earth with great circular ease.

Sieve of
Er-a-tos-the-nes

The famed Alexandrian oft tried,
Finding numbers no others divide.
"Perforations give,
A prime spotted sieve,"
He proclaimed with composites aside.

Other Mathematicians

How thorough his prime searches had been!
Yet acknowledge we must other men,
And women of fame,
With zeal all aflame,
Accepting the deep challenge as then.

Numbers and the *Elements*

The books of the *Elements* contain,
Geometry, both solid and plane.
Still numbers abound,
As Euclid there found,
Every largest prime quest was in vain.

Infinitude of the Primes

'Tis a proof we will hardly forget,
As its logic enthralls our thoughts yet.
"Q.E.D." Correct.
In steps indirect,
All the primes form an infinite set.

Mersenne Primes and Perfect Numbers

One less than a large pow-er of two,
Occasionally prime, Euclid knew.
Again and again,
With primes of Mersenne,
Numbers perfect break evenly through.

Fermat Primes and Euler's Follow-up

Fermat, much conjecturing had done,
When Euler's work fully was begun.
Such forms may be prime,
But not every time,
These pow-ers of two increased by one.

Fermat Primes and Constructibility

The regular pentagon was seen,
In the light of a golden-like mean.
Constructible? Yes!
But the Greeks must guess
Of the figure with sides seventeen.

Gauss and the Primes

Through geometry's vast wonderland,
These constructions continued to stand,
Till answers were heard,
In Gauss's plain word,
With his compass and straightedge in hand.

Fundamental Theorem of Arithmetic

The Fundamental Theorem applies,
To integers regardless of size.
From building block primes,
To choose many times,
'Tis uniquely arithmetic's prize.

Mystery of the Primes

The ancient and modern worlds blending,
With hardest of questions transcending,
O charmed history,
O prime mystery,
This vast set of numbers unending!

Absolute Values in Math

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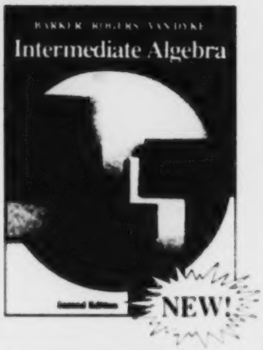

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MATHEMATICS EDUCATION

The Arithmetic Connection

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WESTERN
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Pansy is Coordinator for Developmental Mathematics in the Community College of Western Kentucky University. In addition to teaching, she supervises the Arithmetic Skills Test (AST) Program and regular developmental courses. She received her PhD in Adult Education from Indiana University, Bloomington, an MA in Mathematics Education from Eastern Kentucky University and a BA in Mathematics from Berea College.

The Community College

The Community College, a separate institution under the auspices of Western Kentucky University, was established in May 1986 to expand post-high school educational opportunities in the community and region. The emphasis is on expanding educational opportunities, rather than duplicating existing opportunities at the University. The Community College is designed to prepare students for immediate employment and/or further higher education.

Eighteen associate degree programs and three certificate programs are currently available at the Community College. Courses are offered through five divisions: Business, Health, Technology, General Studies, and Academic Support. It is through the Academic Support Division that developmental courses are offered. Since the Community College is organized as a separate institution within Western Kentucky University, it has a separate faculty, separate programs, and separate courses. However, through a cooperative relationship with Western, many cross-listed courses are taught by faculty from University departments. The Community College office building and classrooms are located on Western's main campus for easy access.

Background

One of the primary purposes of the Community College is to meet the academic needs of underprepared students so that they can proceed to earn a college degree. Students recommended to the Community College for developmental work have ACT enhanced scores less than 20 and a high school GPA less than 2.2 (on a 4.0 scale).

Beginning fall, 1986 an Arithmetic Skills Test (AST) Program was initiated for students in Math 109 – General Mathematics (Math 109 is a survey of topics – sets, logic, probability, statistics, consumer math and geometry). The AST Program runs concurrent with, but in addition to the students' Math 109 class to provide a review of arithmetic. The program includes three versions of a basic arithmetic test with lecture/tutorial sessions between versions to assist students in mastering the content.

Although the first year's assessment of the AST Program revealed that it had achieved its basic goal of removing arithmetic weaknesses in Math 109 students, certain recommendations were made for changes in the program. Two of these recommendations were: (1) offer a bi-term (half-semester) and/or full semester course on arithmetic concepts in addition to the AST review and, (2) track students who enroll in an arithmetic course into the subsequent Math 109 course.

Consequently, a bi-term arithmetic course was begun in spring, 1988. It was taught by part-time adjunct faculty until fall, 1989, when a full-time instructor was hired. With a full-time faculty member to coordinate the developmental mathematics program, a full-semester arithmetic course – Basic Mathematics (Math 050) – was begun and students were tracked into their subsequent Math 109 course work. So, Schwartz's (1990, p.30) ninth question about developmental mathematics – "do we follow up enough on students to see how they fare in later math courses..." – could be answered "yes, we have begun." The purpose of this paper is to share the results of a follow-up study of those students who enrolled in Basic Mathematics in fall, 1989 and proceeded to General Mathematics in spring, 1990.

Basic Mathematics (Math 050)

Math 050 is designed for students who have a weak background in mathematics and plan to take Math 109 for their primary college level mathematics course. Community College students who fit this description are strongly advised into Math 050. Many university students also enroll. The content for Math 050 consists of fractions, decimals, percents, geometric measurements and problem solving. The instructional format is primarily short lecture/discussion. Students frequently work in teams and/or small groups on nonroutine problem-solving assignments. Some of the initial group work is done in class but most is done outside of class after teams are formed by the instructor. The importance of regular attendance is stressed. Homework is assigned and checked daily (either through notebooks or turn-in problems) to monitor students' progress. Quizzes are given regularly (at least bi-weekly) and there are four in-class tests and a comprehensive final exam.

A primary purpose of Math 050 is to review arithmetic so students can pass the AST in Math 109. Please note that not all of the students in Math 050 are taking the course in preparation for Math 109 and, even if they were, these students may not choose to enroll in Math 109 the subsequent semester, although such action is highly recommended.

Table 1 lists pass-fail numbers and percentages for the fall, 1989 Math 050 bi-term and full-semester classes as well as the subsequent AST in Math 109. Altogether 96% (22 of 23) of the fall, 1989 Math 050 students passed the AST in the spring, 1990 Math 109 class.

It should be noted that about half of the students who passed Math 050 in fall, 1989 did not enroll in Math 109 in spring, 1990. Similarly, there were students enrolled in the spring, Math 109 class who had taken and passed Math 050 *prior* to fall, 1989. Thirteen students fell into this category and 10 of them (77%) passed the AST in Math 109. Such a high percentage passing after a hiatus from mathematics speaks highly of the retention level of these Math 050 students.

Table 1

Achievement in Math 050, Fall '89

Course	*n*	# Pass	% Pass	# Fail	% Fail	# Drop	% Drop
Semester	55	39	71%	6	11%	10	18%
Bi-term	29	16	55%	9	31%	4	14%

Achievement on AST, Spring '90

Course	*n*	# Pass	% Pass	# Fail	% Fail
Semester	20	19	95%	1	5%
Bi-term	3	3	100%	0	0%

Table 2 compares the achievement for the review Math 050 students, regular Math 109 students in the Community College and University Math 109 students. The numbers in the review group include the students who took Math 050 prior to fall, 1989. The passing rate for the review students was 5 percentage points higher than that of the other two groups. This achievement on the AST is the main purpose of offering a Basic Mathematics course and directing appropriate students to it before Math 109. It should be noted that the regular Community College students (those not needing a review of arithmetic) did as well on the AST as did University students.

Table 2

Achievement on AST for Three Groups

Students	# Pass	% Pass	# Fail	% Fail
Review CC	32	89%	4	11%
Regular CC	26	84%	5	16%
University	481	84%	95	16%

Achievement in Math 109

Table 3 displays the grade distributions for the Math 109 students in the Community College sections. The "W" category includes those students who withdrew, audited, or received a grade of incomplete.

The main differences between these two groups of students is a higher percentage of A's and B's in the regular group and a higher percentage of C's in the review group. Such a finding is not surprising once we are reminded that those students needing a review of arithmetic most likely will have more difficulty with college level concepts than those students not needing such a review. Otherwise, the Math 109 pass/failure rate (withdrawals included) is quite similar for both groups, 55%/45% for the review group and 52%/48% for the regular group. Thus, it appears that those Community

College students who took care of their needed review in arithmetic, proceeded to achieve in Math 109 at a comparable level with regular Community College students.

Table 3

Grade Distribution for Community College Math 109 Students

	Review Students		Regular Students	
Math 109 Grade	# Students	% Students	#Students	% Students
A	2	7%	3	10%
B	1	4%	5	16%
C	7	26%	3	10%
D	5	18.5%	5	16%
F	4	15%	6	19%
W	8	29.6%	9	29%

Conclusion

First, the main purpose for offering a Basic Mathematics (050) course has been realized. Students who need and take Math 050 proceed to do as well on the AST as students not needing a review course. This author agrees with Don Cohen, (1989, p.v.), that "when colleges enroll students without making extensive efforts to provide the necessary educational support so that all students have a 'realistic' opportunity to be successful, we might as well adopt Kelley's (1989) position and not enroll high risk students in the first place."

On the other hand, Kelley (1989) rightfully asserts that zero grade-point averages can be the result of chronic non-attendance and incomplete assignments by students. Policies of regular attendance and homework are strictly enforced in the Math 050 course in the Community College classes at Western. On the first day of class, students are informed through a handout that three absences may result in their dismissal. In short, this follow-up study lends support to the argument that the Community College, with its smaller class size and greater individual attention, is assisting the academically weak students in preparing themselves for and having success in college level work.

Readers are invited to contact the author for further information on the programs described in this article.

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College Placement Testing: A Local Test Alternative

by William L. Truman
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William L. Truman received his PhD in Mathematics Education (1985) and his MS in Mathematics (1974) from North Carolina State University. He earned his BS in Mathematics from Pembroke State University in 1969. He has taught at the junior high, high school, community college and four-year college levels.

A recent issue of *Focus* (MAA, 1990, March/April) cites mathematics placement as a major concern of many colleges and universities. Students coming to colleges need courses ranging from arithmetic through calculus and the proper placement of these students is critical.

In 1985 Pembroke State University recognized the growing need for accurate placement and began to revise its placement procedures as part of an overall college entrance testing program. Pembroke is a public four-year college located in Pembroke, North Carolina. The enrollment is approximately 3,000 students.

While there are commercially prepared testing programs available, we decided to develop our own test because the scores are a direct measurement of student skills taken under conditions that are uniform for all students, and data can be accumulated that verifies the effectiveness of each question on the test. In addition we decided to prepare one test for students at all achievement levels so that it could be administered during summer freshman orientation. Having the results of the placement test available before the students register for their classes has proven to be most useful for our students.

The particular placement test we have developed is valid only at Pembroke. However, the procedures for test construction, for test validation and improvement, and for collection and use of data can be generalized to larger populations in both colleges and high schools. Five years of close work with the test and extensive feedback from administration, faculty, and students have established the test as a reliable instrument. A report of this experience should provide valuable information and insights for others.

Features of a Good Placement Test

Features of a good test identified in the literature guided our planning and implementation of the placement test.

Carefully Constructed and Selected Questions. Questions on a placement test should test for or measure the prerequisites a student needs in order to be successful in a course. A common error in the initial selection of questions is that test makers construct questions appropriate for a final examination in a course (MAA, 1984). We tell our faculty to consider questions appropriate for the first test since most courses begin with a review of fundamental concepts needed to master the subsequent material.

In order to increase the likelihood of a greater variety and larger number of test questions, it is important to involve as many faculty members as possible in submission of questions. Stiggins and Bridgeford (1985) suggest that the more faculty involved in test item construction, the better the overall quality of the test. Our experience indicates that ten initial questions may result in two questions with suitable statistics for test use. As Miles (1981) states, "No a priori estimate of how an item ought to work can substitute for the measurement of how it actually does work with real students" (p. 3).

Multiple-choice questions may begin as open-ended questions that are similar to those used in a typical mathematics test. Possible distractors can be chosen from the students' incorrect responses that indicate a misunderstanding of the concept and occur frequently enough to suggest a common error. Experienced teachers also will be able to provide a good list of distractors. The Educational Testing Service suggests that if a distractor is not chosen by anyone in a large sample (100 or more), it should be replaced (Katz, 1965). In terms of the number of distractors, a study by Budescu and Nebo (1985) indicates that the number of options should be greater than three, and that the optimal number is five for low ability test takers.

After a question is constructed, it must be tested before it is used for evaluation purposes. During this time we call such a question an alternate item. Each year faculty members teaching the targeted courses are asked to generate at least five test items to be considered as new placement test items. These faculty are asked to examine the test items presently used for placement in (or out of) their particular courses and to create new items based on different aspects of the course content. The newly-generated items may be in either open-ended or multiple-choice format. During the semester these new items are given as a short test to students presently enrolled in one of the entry-level math courses. The discrimination index and the difficulty index for these items are then computed. (These terms are explained later in the paper.) The number of correct and incorrect responses for each item is compared with midterm grades to indicate the potential that item has for positive correlation. Table 1 illustrates a typical comparison.

Table 1: Indicator of a Question's Potential Correlation

Midterm Grades	F	D	C	B	A
Number of Correct Responses	3	4	11	9	6
Number of Incorrect Responses	7	5	12	5	1

The placement test committee then selects the best two or three items from each level for testing as possible test items. The next year's placement test contains twelve to fourteen of these items. The alternate items are not used to place students the initial year, but are used as a source of proven items for future tests. **An item on the test is not replaced unless the content of the item is no longer considered appropriate or an**

alternate is considered to have better statistics. This principle mandates that each item be tested with two different groups of students before it is included on the placement test.

Appropriate Norms. When selecting questions for a placement test for a particular course, some content areas (e.g., functions) are appropriate for several sections. However, the level of difficulty (percentage of students getting the correct response) of the question can be used to identify the section for which the question is most suitable. We arrange the questions on the test into sections and arrange the questions inside each section according to difficulty. A goal in ordering the test questions is that a student will be able to work almost all the problems before his/her level of placement and few afterward (Miles, 1982b).

The level of difficulty appropriate for a particular section is a local decision. We have found that evaluating the questions for difficulty is a particularly effective means of selecting potential questions.

Within a section of questions the cut-off score, the number signifying passing a section, is also basically a local decision. The MAA User's Guide (1984) notes that attempting to use national or regional data is rarely effective. We adjust our cut-off scores according to prior students' performances. For example, if the cut-off score for a particular section is 9 and most of the students taking the course who scored 9 made an A or a B in the course, we may lower the score the following year. For the initial year we suggest that students be placed into classes according to some departmentally defined quotas. We used the ratios based on the size of classes from the previous year.

A good length for a placement test is from 25 to 50 questions (Miles, 1982a). The reliability of any test can be improved by increasing the number of items of the same quality (Diederich, 1973). However, the number of items a student can answer before fatigue seriously compromises ability limits the number of similar quality items which can be used. In our experience, students with low ability tend to tire more quickly than those with high ability. To balance these concerns, items and levels are ordered from the least to the most difficulty. Our test is somewhat long – 56 to 58 questions; however, most students do not complete the entire test. The students are told with their beginning instructions that the questions are arranged in order of difficulty and that they can turn the test in when the questions have become too difficult to answer. The average student answers 35 questions, and only the most able students complete the test. In general, most students turn in the test within one hour.

Suitable Standards. The Standards for Educational and Psychological Testing (1985) suggest the following standards for measurement to be met.

1. **Content validity.** Content validity deals with the suitability of items chosen from each content area. Content validity is established by a mathematics department committee consisting of at least three and at most six members. Content validity is verified before any other standards are checked. In order for an item to be considered content valid, it must satisfy at least one of the following:

1) it must test subject matter taught in the level of testing, or

2) it must test previously learned knowledge or a subskill needed to be successful in the course.

Content-related evidence of an item is collected each year the item is used on the placement test (Anastasi, 1976). Occasionally, an item with good predictive validity will be removed from the test because course content has changed.

2. Correlation. Once content validity has been checked, a correlation of the placement test with some measure of success is determined. The placement test committee compares students' final grades in initial mathematics courses with their performances on the corresponding battery of placement test items. This procedure was formulated after discussions with students, faculty, and administrators. Hasset and Smith (1983) noted that their experience with correlations of students' scores on placement test batteries and final grades usually led to correlations with the course grade in the range from 0.35 to 0.50. The MAA User's Guide (1984) notes that even in excellent placement programs, correlations between course grades and placement test scores are weak. It should be noted that when the number of students used in the comparison is small, correlations from year to year may vary widely. In addition, models developed on small samples tend not to be as consistent or reliable as those on large samples.

3. Difficulty. To set a standard for a proper level of difficulty (percentage of correct responses), the committee decided that on average, 75% of freshmen students should be able to answer the questions designed for the least difficult level while 25% of the students should be able to answer questions for the most difficult level. The committee established a sliding scale of ideal difficulty indices for each course between these extremes. The selection of these percentages was based on the committee's knowledge of our students and courses. This type of decision should again be a local decision, using national norms or other school data only as a guide.

4. Discrimination. Discrimination is the difference between the percentage of students in the upper quartile getting an item correct and the percentage of students in the lower quartile getting the same item correct. For example, if 50% of the upper quartile answered a particular item correctly and 20% in the lower quartile answered the same item correctly, this item's discrimination index would be 0.3. Diederich (1973) suggests a minimum discrimination index of 0.15 for most items. Exceptions can be made for very easy or very difficult items. An item on which 90% of the students respond correctly may be a good "starter item," but it will probably have a low discrimination.

5. Reliability. Diederich (1973) defines reliability as "an estimate of how close you would come to the same set of scores if you gave a parallel form of the test" (p.10). Two methods of measuring reliability are used. The first is the split-half correlation (Spearman-Brown formula). Katz (1973) suggests that for teacher-made tests a 0.6 split-half correlation is the minimal acceptable scores with an expected range of 0.6 to 0.8. Katz notes that when tests are being considered for publication with national norms, test publishers generally regard 0.9 as a minimal value.

The other measure of reliability is the number of students who believe the placement test did not accurately measure their mathematical skills. In the last three years, less than 2% (23) of approximately 1,375 students have asked for a retest. Of those 23, only seven earned scores which placed them in a higher level course. Conferences with the dissatisfied students have led to valuable suggestions and modifications for our placement test.

6. Validity. Anastasi (1976) defines validity as "what the test measures and how well it does so" (p. 134). According to Standards for Educational and Psychological Testing (1985), the two errors to avoid in a predictor test are "false positive – selecting someone who will subsequently fail," and "false negative – not selecting people who would have succeeded" (p. 11). According to the MAA User's Guide (1984), "the validity of . . . [any placement test] . . . will necessarily depend on the content and the prerequisites of the courses into which students are being placed. They are, thus, best measured by user mathematics departments" (p. 10).

Procedures

The following discussion of procedures represents a summary of what actually worked on the initial test and how we have built on our successes. As should be expected, not everything we tried worked.

In designing the placement test, we were seeking information which could be collected from the students before they select their first courses and could be analyzed, tested, and changed based on its suitability for students and courses at Pembroke State University.

During the first year of placement testing, we followed these steps: (1) The five sections of placement which closely match the entry-level math courses offered were identified. These are Mat 104 – Pre-Algebra (mainly arithmetic), Mat 105 – Introduction to College Mathematics (beginning algebra), Mat 107 – College Algebra, Mat 108 – Trigonometry or Mat 109 – College Algebra and Trigonometry, and Mat 221 – Calculus; (2) Items appropriate for each course were selected, tested, and revised.

Each year the test is evaluated for needed revisions and modified with careful adherence to the standards previously discussed.

Recommendations

Based on our experience and study, the following advice should be helpful for placement test construction. Noted parenthetically are other studies supporting our recommendations. Supporting studies concerned with multiple-choice testing rather than placement testing are also included.

1. The standards established by the most recent Standards for Educational and Psychological Testing should be available as a resource for the placement test committee.

2. Content validity should precede all other evaluations (Anastasi; Bloland and Michael, 1984) and should be re-evaluated each year.

3. Statistical methods rather than subjective opinions of teachers should be used to determine predictive validity (Miles, 1981). This approach is especially important for items requiring higher level thinking skills (Coladarci, 1986). In general, well-constructed items (those with good discrimination and well-chosen distrac-

tors) are more consistent predictors than past items which have merely good correlations to success.

4. Data should be collected over an extended period from large data samples in order to establish a test with good predictive ability (Miles, 1982b).
5. In placement test construction it should be remembered that validity for lower level assessment is more difficult to establish than for higher levels (Coladarci; College Entrance Examination Board, 1986).

Discussion of Our Data

During the five years of testing, the data collected have been consistent. This indicates that once a question has been shown to have "good statistics," the statistics for that question tend to remain good in the following years. Consistency of good statistics is also noted by Waits and Demana (1988) in their discussion of the use of placement tests at Ohio State University. Also, we have found that replacing an item only with a statistically better one seems to improve the overall results each year. For the last two years, the split-half correlation of the tests has been .856 in 1989 with 487 students and .877 in 1990 with 475 students.

As a measure of the false positive test – that is, a predictor test which predicts success for someone who ultimately fails – Table 2 shows that out of 461 students taking the recommended courses in the 1988-1990 fall semesters, 53, or 11.5%, subsequently failed.

Table 2: Comparison of the Placement Level of Students with Their Final Grades for the Fall Semesters 1988, 1989, 1990

Number of students taking a course:	Final Grades					
	A's	B's	C's	D's	F's	W's
Below the placement level	52	78	57	42	21	12
At the placement level	58	112	136	81	53	21
Above the placement level	5	6	13	16	10	2

Another judgment focuses on avoiding the false negative – that is, placing a student lower than his capabilities. While there is not a direct measure of this phenomenon, Table 2 provides some statistics for this measure. When students take courses lower than those recommended, they generally do well. It should be noted that a placement test measures the ability of a student to perform mathematics at a certain level, not the requirements of his or her major. For example, English majors are not required to take Calculus; a student who intends to major in English may elect to take a lower level course even though he or she places into Calculus.

Conclusion

After five years of mathematics placement testing at Pembroke State University, the mathematics department is convinced that this program provides an efficient, practical, and workable method of placing students in mathematics courses which give them the best chance for educational success. The placement test is neither a diagnostic instrument nor a final comprehensive examination; rather, it is a guide, based on the past performances of other similar students, of potential success in a specified set of mathematics courses. While the test cannot measure motivation or other affective variables in mathematics learning (Boamah, 1986), it can go far to provide our students with a healthier and more valid mathematics experience.

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Educational Reflections

Writing in Mathematics

by Deborah A. Crocker
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"Writing across the curriculum" and "writing to learn mathematics" are both phrases that are heard frequently today. Are they the same? If not, how are they different? This article is an attempt to clarify the meaning of writing across the curriculum and writing to learn mathematics. Also included is a look at some of the ways to use writing in the mathematics classroom with comments on the possible effects on the teaching and learning of mathematics.

Introduction

Writing across the curriculum is a movement that has gained much attention. While the phrase is used in numerous ways, the goal of the movement is to improve students' writing. Alternately, the terminology "writing to learn mathematics" is reserved specifically for attempts to incorporate writing into the mathematics classroom with the purpose of improving students' understanding of the mathematics (Miller & England, 1989). Yet, Toby Fulwiler (cited in Kenney, 1990) of the writing across the curriculum movement, when asked to explain the difference between the two said, "Writing Across the Curriculum is a synthesis of writing and learning – that we do more learning by doing more writing and that we become better writers by being better learners" (p. 17). This tends to indicate that the two phrases are inextricable. Thus, perhaps we can use them interchangeably. As mathematicians and mathematics educators, we are most concerned with the goal of improving mathematical understanding, whatever terminology we employ.

The National Council of Teachers of Mathematics (NCTM) in the *Curriculum and Evaluation Standards for School Mathematics* (Commission on Standards for School Mathematics, 1989), specifies mathematics as communication as one of the four standards included at all grade levels. As part of their goals for students, NCTM lists, "...that they learn to communicate mathematically" (p. 5). This mathematical communication is further explained by:

> The development of a student's power to use mathematics involves learning the signs, symbols, and terms of mathematics. This is best accomplished in problem situations in which students have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. As students communicate their ideas, they learn to clarify, refine, and consolidate their thinking. (p. 6)

Writing in mathematics is an integral part of learning to communicate mathematically. This includes the appropriate use of mathematical symbols and the ability to write about mathematics using the proper vocabulary, punctuation, and spelling. By writing, students construct meaning and understanding. As mathematics teachers, we have been too willing to overlook grammar, spelling, and punctuation.

Types of Writing

Two basic approaches for incorporating writing into the mathematics classroom are used (Sipka, 1990). One approach is to structure the writing very carefully with specific goals and guidelines. This approach is often used when asking students to write proofs or explain problems they have solved. This is necessary when a formal product is desired. The second approach is more informal and is often employed to determine the writer's mathematics background, what the writer is thinking, how they feel about the mathematics they are learning, and where questions exist.

An example of a formal approach is that taken by Price (1989). Price emphasizes writing in his mathematics courses by requiring students to write the solutions of problems following very specific guidelines. He comments on students' abilities to write by saying, "nobody asks them to write, so naturally they can't" (p. 393). He further states, about our system of teaching mathematics:

> Our system trains students to go home, zip through their ten or twenty differentiation exercises as quickly as possible, tear the paper out of their spiral notebooks, and hand it in with a sigh of relief. I try to foster a different attitude: that written homework is an exercise in learning, organizing, and explaining, not a certificate of drudgery. (p. 394)

Price expects his students to write and submit drafts of homework problems in much the same way a mathematician writes and revises drafts of articles for publication. In addition to problem solutions, other formal forms of writing include proofs, summaries of articles, and research papers. Any of these can be included in mathematics courses.

Informal writing assignments may be introduced using problems from the teacher to trigger student thoughts. For example, in a study of an elementary algebra review class at the college level by Ganguli (1989), three to five minute writing exercises were used to obtain information on students' understanding of content. Problems for the writing included asking students to: (a) state a word problem to describe a given situation or numerical set of data, (b) describe the most important concept from the last lecture, and (c) give a definition of the most important concept of the day. The assignments were not graded, but used to gauge understanding and to give students the opportunity to express themselves mathematically. Forty-four percent of the students who were given writing assignments obtained grades of A or B compared to thirty-four percent of students in a similar course who did not write. The failure rate was also lower for the class that participated in writing. Other formal methods of incorporating writing into mathematics class include math autobiographies, usually used at the beginning of courses to obtain information on mathematics background; logs, diaries, or journals to record feelings, questions, answers, and interpretations of the mathematics being studied; and letters to the teacher or fellow students discussing the mathematics just studied. Informal writing assignments are often nongraded.

Miller (1991) states that, "students who don't ask questions are willing to write" (p. 517). Writing can be used to identify math anxious students, to determine content that may be causing difficulty, and to discover misconceptions developed by students.

Conclusion

Specific examples of writing in mathematics can be found in almost any professional journal. Some suggested articles to illustrate the use of writing at all grade levels are Haven's (1989) approach to writing in a general mathematics class; Ford (1990) who describes a writing approach used in Grade 3 to enhance the teaching of problem solving; and Cappo and Osterman (1991) who share several examples of writing at various grade levels including logs, journals and problem creating. Those with a strong interest in writing to learn mathematics should read the Mathematical Association of American (MAA) publication *Using Writing to Teach Mathematics* (Sterrett, 1990). Several of the ideas presented here are from papers included in this MAA publication.

Writing to learn mathematics appears to be successful in fostering understanding and enjoyment of mathematics. Research into the effects of writing will need to be conducted to provide evidence that learning is enhanced. Along with an improved understanding of mathematics comes a more positive attitude toward the subject. Writing is a nonthreatening method for evaluating student understanding and answering questions or concerns. Formal writing to improve learning and promote thinking as well as reflection on problems and solutions can only be perceived as positive. Perhaps we should all consider including writing as a more integral part of mathematics teaching and learning.

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Mathematics: An International View

Edited by Igor Malyshev and Joanne Rossi Becker

San Jose State University, One Washington Square, San Jose, CA 95192

This column is as an avenue for discussion of the first two years of college mathematics with a global perspective. Because mathematics is an international culture, it is important to follow the development of mathematics education throughout the world. We feel that by sharing information from other countries with systems of education different from ours, we will gain an enriched understanding of, and a better perspective on, our own system. A completely different viewpoint on a topic can serve to stimulate our imaginations and help us find new solutions to a problem which will work in our context. As we move toward dramatic changes in both the curriculum and instruction in mathematics called for in many reports, reflection on practices used in other countries can only help our efforts.

For this column, we would like to publish two types of items: material with an international origin or context that you could use directly with your students; or information about the content and structure of coursework in other countries. We welcome submissions of 3-4 pages or suggestions of mathematicians from other countries from whom we could solicit relevant material.

Send contributions for this column to Igor G. Malyshev at the address above.

Mathematics Education in Botswana

by John Mitchem
San Jose State University, San Jose CA 95192

John Mitchem spent the 1990-1991 academic year as a Fulbright Lecturer at the University of Botswana. He has been at San Jose State since completing his PhD at Western Michigan University in 1970. Before going to graduate school he taught high school mathematics in New Jersey and in Tanzania, East Africa.

Running from a charging elephant on an island in the Okavango Delta, observing a pack of wild hunting dogs near camp in the central Kalahari, and watching the African sunset on the Chobe river are images tourists remember from their Botswana visit. Botswana, however, is not only a country with wonderful wild animals, vast stretches of open space, and ancient Bushman rock paintings. It is also a country working hard to use its new mineral wealth to modernize and to prepare for the 21st century.

Botswana, a Texas-sized country of 1.3 million people, became independent in 1966. At independence when it changed its name to Botswana, Bechuanaland Protectorate had about 1550 students in its 9 secondary schools and only 40 or so citizens who were university graduates. Today this country which shares a thousand-mile border with the Republic of South Africa, has about 125 secondary schools enrolling some 50,000 students in grades 8 to 12. At the tertiary level there are 5 or 6 teacher training colleges and the University of Botswana with affiliated institutions, the Botswana Agricultural College and the Botswana Polytechnic. The University currently enrolls about 3300 students.

Remarkable educational progress has clearly been made in the two and a half decades since independence. How has such prodigious growth in educational opportunity been financed, given the fact that in 1966 Botswana was one of the very poorest countries of the world with a per capita annual gross domestic product of less than $100? The answer: diamonds! Since independence, Botswana has opened 3 major diamond mines which have been the driving force in making it one of the fastest growing economies in the world and propelling the per capital annual gross domestic product to over $2100. A significant amount of this new wealth has been used to expand the educational system. Furthermore, Botswana's stable democratic government has attracted significant education aid from countries in Western Europe and from the United States.

Yet despite the huge increase in educational opportunity, Botswana still relies heavily on expatriates in many science and mathematics based jobs. Perhaps the most pronounced example of this shortage of scientific personnel is the Mathematics Department of the University of Botswana. In May 1991 the Department had 16 faculty members, none of whom were citizens of Botswana. Since Botswana's diamonds won't last forever, a major concern is how to keep the development engine moving ahead as diamond revenues stop growing and eventually decline. This is particularly important given that the population is growing about 3.7% per year.

In response to both the current shortage of technical citizens and in the hope of developing a long-range future that doesn't depend so heavily on diamonds, Botswana is placing a heavy educational emphasis on mathematics, science, and technology. This emphasis is particularly noticeable at the University.

Students are accepted to the University only if they have performed well on their "Cambridge Examination," a series of very comprehensive exams, equivalent to England's "O" levels, taken at the end of the 12th year of education. Virtually all citizens who qualify for the University have all of their expenses paid by the government. Those who have done well in mathematics and science are strongly encouraged to enroll in the Faculty of Science. In order to help students succeed at U. B. in math and science as first year students, the government sponsors a special Pre-Science Program. All new Faculty of Science students are given 28 weeks of instruction at the University before they begin their four years of university work. The Pre-Science Program has its own faculty of 20 to 25 teachers to serve the 300 students who take courses in math, physics, chemistry, biology, computer science, and English. At the end of the 28 weeks most of the students pass the examinations required to enter the Faculty of Science as first year students. The fact that no U.B. students except those in Science have this extra prep for their university work indicates the Government's serious commitment to science and technology.

Setswana is the first language for most citizens. But by grade 5 all instruction except for the Setswana class is in English, and subsequent exams are in English. For Science students their last English course is in the Pre-Science program. Once enrolled in the Faculty of Science at U. B. all of their courses will be in mathematics and science unless they leave the Faculty. Virtually all first year students take courses in math, chemistry, physics, and computer science. All but the last are full year courses; computer science is one semester only. Many students also take a year-long biology course. First year science students spend as many as 35 hours per week in class, labs, and tutorials.

The first year mathematics course is a fully integrated course covering topics from college algebra, trigonometry, analytical geometry, and calculus with a significant number of quite difficult problems but relatively little theory. The students are in general better at manipulative algebra than all but the best U.S. students. About one-fourth of the way through the one year course students who are having difficulty in mathematics are shifted to a somewhat less difficult course which covers many of the same topics but at a less strenuous level. Such students are then forbidden from majoring in mathematics or physics, and select their major subjects from chemistry, biology, geology, and environmental science.

Third and fourth year students have all of their courses in one or two disciplines. For example, many math students study physics as well, and all of their courses are in mathematics and physics. Also at the end of the first or second year some Faculty of Science students decide to become teachers. They take half of their third and fourth year courses in the Education Faculty. These courses include a number in the Mathematics Education Department as well as a number in other departments of the Faculty of Education. The other half of their curriculum is regular courses from the Mathematics Department. Although U. B. does not have a unit system which is so common in the U.S., third and fourth year math students doing another science such as physics, or doing education take roughly the equivalent of 9 units of mathematics each semester. Mathematics has recently introduced a "single subject" for students who want to study only math during their last two years. A student has to perform significantly better than average to be accepted as a single subject student. One of the ironies is that mediocre and weak students must study 2 subjects roughly equivalent to 18 units per semester whereas an outstanding student who selects the "single subject" option takes only 15 units each of his/her last 4 semesters.

Grading is very strict with the grade for each class largely determined by a difficult final exam given at the end of the academic year – even for courses which were taught only in the first semester. A grade of 80% is required for an "A." A grade of 40% is required for a limited pass in a course. Nevertheless, students are either ruled as passing the entire year's work or as failing. In the latter case the student is usually allowed to repeat the year's work, otherwise he/she is dismissed from U.B. After exams have been marked by both the instructor and the external examiner the Department meets to discuss grades for the students. Next, all instructors in the Faculty of Science meet to assign final grades for all students for the year. At this meeting it is determined which students will be allowed to take supplemental exams two months later in an attempt to raise their grades to a pass. There is absolutely no grade inflation in the Faculty of Science. For example, of the 900 or so course grades given in May 1991 to first year Science students, 5 were A's. (Yes! Only 5 out of 900.) Since 4 of those were in computer science, the instructor in the course was required to explain to the Faculty why so many students received A's! In 1991 the pure mathematics examiner was from the University of Zimbabwe and the applied mathematics examiner was from Manchester University. External examiners are charged with insuring that "international standards" are upheld.

The second, third, and fourth year courses tend to be very similar to courses in the U.S. For instance, during my year at U.B. I taught first semester modern algebra to third years and graph theory and combinatorics to fourth years. Both courses were essentially the same as courses at my home university. One special course very appropriate for Botswana but not common in the U.S., is a course in wildlife modeling. Also, with the recent introduction of the single subject major, the undergraduate course offerings in Mathematics are much deeper than at most U.S. institutions.

Students can take a four-semester sequence in modern algebra and a two-semester topology sequence, for example. As of now, no student has completed these sequences, but the University expects to double its enrollment in the coming decade.

Many of the country's secondary mathematics teachers have completed a two or three year teacher education program after secondary school. Each year the government sends 10 or 12 of these teachers to U.B. for two years to complete a degree in math education. Their studies are split equally between Mathematics and Education. This program is very reminiscent of the National Science Foundation programs in the U.S. in the 1960's. The teachers take special math courses which recognize their five year absence from formal study and are perhaps more applicable for teachers than the usual third and fourth year curriculum. I taught a one-year course in advanced Euclidean geometry to 5 of these experienced teachers. Geometry is still an important portion of the secondary curriculum. Probably the most important difference in secondary mathematics from the U.S. is that mathematics is an integrated subject. Students study mathematics rather than algebra, trigonometry, and geometry.

Botswana is a small country that is still impoverished by Western standards. It is, however, using a nontrivial portion of its recently generated wealth for mathematics, science and technology education in the hope of entering the 21st century with the ability to sustain economic growth without the current heavy reliance on diamond mining. I was fortunate to be able to teach for a year at the University of Botswana and wish the government and the people success in their challenging endeavors.

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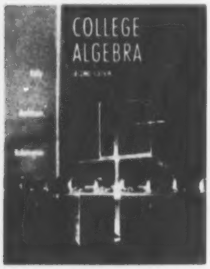

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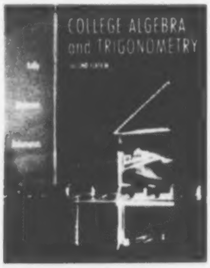

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Pandora's Rectangular Parallelepiped

Edited by L9882 × 11 $\int_0^{NER} SEC^2x\,dx$

Pat Allaire of Queensborough Community College (Bayside, NY) and Theodora Tsakiris of GateWay Community College (Phoenix, AZ) were the first to respond with their decipherings of the "pictel" which appeared in this column in the Spring, 1991, *Review*. They explained that it clearly showed a square root and a cube root of two. It has also been suggested that the plant itself might be a binary tree, but far be it from me to know what the pictel's creator Ellartee had in mind.

Along the same line as the pictel are the figures at the right. I have heard them referred to by several names, but the person who sent in these two examples calls them *wuzzles*. In any case, it won't be hard for many of you to figure out the meaning of the top wuzzle, because it describes someone with whom you are, unfortunately, intimately acquainted. They were sent in by Russ Blankenfeld, of Rochester Community College (Rochester, MN).

PAID
TEACHER
WORKED

CHIFF
π

It had been hinted at in an earlier column, but who could believe that it would actually happen?

The Perfessor is retiring![1]

Of course, he (perhaps I should say He) took with him most of the things which he had accumulated in XXXIX or XL years of teaching, but he did leave behind in his office a number of artifacts. Since you readers have become somewhat acquainted with him (Him?) over the past several years, I thought you might be interested in what some of those things are.

- A well-used typewriter correction ribbon (which forever lays to rest The Perfessor's claim to perfection in all matters, mathematical or otherwise.) I am going to hang on to this one. There may yet arise occasions upon which it wouldn't hurt to remind him of some of the words and phrases he had at one time or another removed from a memo he was writing to the Dean.
- A large box containing such things as used staples, little round pieces of paper from the departmental hole punch, and hunks of chalk too small for a chalk holder. (At one time, he had on hand several hundred pounds of chalk dust, believing that it

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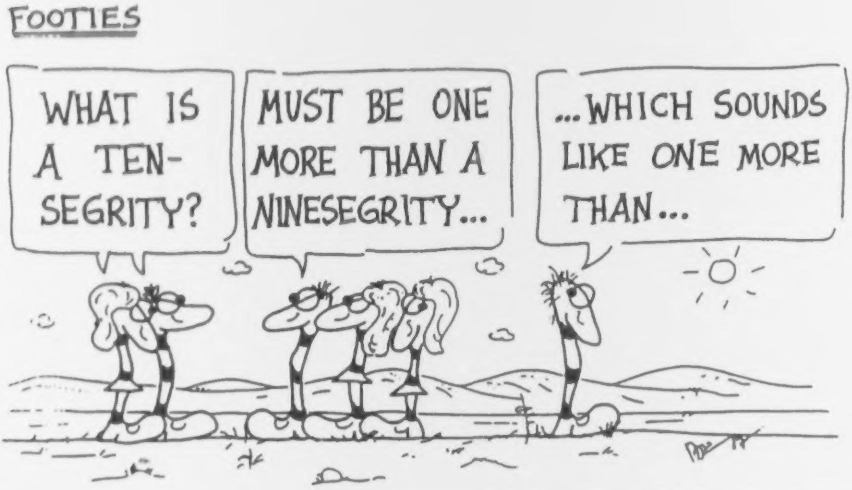
[1] **Editor's Note**: The character of "The Perfessor" is based on a very good friend and colleague of humor columnist Les Tanner. The Perfessor is retiring because Les' tenure as the *AMATYC Review* humorist is concluding.

was only a matter of time before someone invented a chalk-recycling machine. However, upon being reminded that this chalk dust was the residue from his many years of lecturing and that by merely re-associating the bits of dust as they originally appeared on the blackboard his lectures could be resurrected, he removed it to a permanent storage area down the street.)

- A master list of the files in which he kept the lists of all of his index cards. (The Perfessor was **organized**. He was, in fact, the only person I know whose dictionary has a complete index.)
- A base-1 abacus, upon which he purportedly did so much of his fundamental and exhaustive work (resulting in 7 volumes, unabridged) on the properties of 0. [In case you missed that one, a base-***10*** abacus has ***9*** beads on each reed. LRT]
- Last but certainly not least, a copy of his research into the little known (and understandably so) archeological site known as **Snowhenge**. This was apparently constructed in the same spirit as was the famous Stonehenge, partly as a device for determining when Spring had arrived. There is evidence that it was necessary to reconstruct Snowhenge on an annual basis, since its location was somewhat below the 45th parallel. The Perfessor has not yet formed a conjecture concerning precisely what characteristic of Snowhenge permitted its designers to conclude that Spring had indeed arrived.

I would say that The Perfessor will be missed, but the fact of the matter is that he still wanders in at 7:59:30 AM on the button, and has to be re-directed to his new office.

The following *Footies* cartoon from Andrejs Dunkels was inspired by the strange structures exemplified by the photo of the Expanded Octahedron which appeared on the cover of the Spring, 1991, issue of the *AMATYC Review.*


FOOTIES
WHAT IS A TEN-SEGRITY?
MUST BE ONE MORE THAN A NINESEGRITY...
...WHICH SOUNDS LIKE ONE MORE THAN...

Did you figure out the meanings of the wuzzles?

Underpaid and overworked teacher.

Chiffon pie.

The Chalkboard

Edited by
Judy Cain and Joseph Browne
Tompkins Cortland Comm. College
Dryden, NY 13053
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Syracuse, NY 13215

This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Send your contributions to either editor.

Student Referees

In a sophomore level course required of all math majors at my school, a secondary, but important, goal is to develop students' ability to read mathematics. Toward this end I have, in the past, assigned brief sections from upper level texts to be read and paraphrased. The results were disappointing, largely because the material was either too difficult or uninteresting for the students.

While contemplating ways to improve on this, I received a manuscript to review for this journal. It contained material on a subject familiar to most of the students and included proof techniques we had been discussing. I gave copies of the article to three of my best students and asked them to read it and write any appropriate responses in the margins.

The remarks of these "student referees" were interesting, encouraging, and helpful. One paid a great deal of attention to detail and found several mathematical and typographical errors. Another read the paper in a broader perspective and noted how one of the theorems reminded him of similar results in calculus. The remarks of the third student (an English minor) focused on clarity and included several suggestions on how to improve the article's readability.

In retrospect, this exercise had several important benefits. First, the students gained confidence in their ability to read mathematical literature because each was able to recommend at least one improvement in a professional (yet understandable) paper. Second, each was able to see an "application" of our work in class. Finally, the opportunity to offer suggestions and raise questions gave them a sense of making a contribution to the mathematical community. This is not just an illusion; their remarks made for a more thorough review than might have occurred otherwise.

Next semester, I plan to solicit comments from the entire class on an article, and will experiment with methods of having students report their critiques to the class.

Submitted by Raymond McGivney, University of Hartford, West Hartford, CT 06117

Know Thy Calculator

After watching my first year business mathematics students tackle a problem which needed the use of a calculator, e.g., $7000(1 + 5\frac{1}{2}\%)^{-3.5}$, I concluded that they really did not utilize the calculator's full potential. Students were writing down intermediate results of additions and multiplications, etc., instead of using the memory button or the parenthesis option. Essentially they only used the +, −, ×, + , and = keys.

I decided to have a "Getting to Know Your Calculator" class. The students were asked to bring their scientific calculators and instruction booklets to class. The latter demand seemed to puzzle some students; who ever heard of reading instructions before using a calculator? Problem sheets were handed out with exercises similar to

$$200\,e^{-0.5},\ 500\left(\frac{1}{4.21237}-6\%\right),\ \text{and}\ \frac{500[1-(1+(7/12)\%)^{-360}]}{(7/12)\%}.$$

I worked a few examples step by step with the students following. Since there were several makes of calculators, we had numerous stops to find the "second function" key, the "inverse" key, etc.

In forty minutes students were manipulating complex fractions with relative ease (and excitement). The students collaborated among themselves and with the teacher. Those with similar calculators were checking results and sharing warnings.

I felt the class time was spent productively; it prevented many interruptions in later classes and saved the students a great deal of frustration. I received many favorable comments about the class, especially from the non-traditional students. For myself, I enjoyed the class and have hope that the students will keep on reading the instruction booklets!

Submitted by Nivan Saada, University of Maine, Bangor, ME 04401.

Editor's Note: Some previous contributions to this column have mentioned random and student generated components in testing. Here are a couple other ideas we have received along those lines.

Random Homework Collection

Sometimes it seems that students only do homework when it is to be collected, but collecting homework all the time can be overwhelming. To improve on this situation I have started the following random method of homework collection. On the day that an assignment is due a student rolls two dice. If both turn up even, the homework must be turned in. Otherwise, my solutions are posted for the students to use to check their work.

It is easy to change the proportion of the time that homework is collected by changing the condition for the dice. With this method the students (and I) don't know ahead of time whether papers will be collected, so they have to be ready at all times. This has helped my students keep up to date; the questions I get during office hours are on current topics instead of what we did two weeks ago.

Submitted by Roberto Bencivenga, Red Deer college, Red Deer, Alberta T4N 5H5.

Self-selected Test Questions

Test anxiety can be lowered by having some of the items on the test be written by the student who is taking the test. Furthermore, writing the questions provides a review of the material, emphasizing "What do I know about this topic?" Here is how I handle this.

One class period before the test each student turns in a set of questions related to the topic being tested. These questions may be taken from the text, the class notes, or be "originals" by the student. On the day of the test I return this set to the student with a subset of the questions highlighted to be completed as part of the test. The rest of the test, written by me, is the same for all students.

A very real concern for this method is "grade inflation." I have found that it does tend to raise scores for a high percentage of students. However, an analysis shows that there is little difference between students' ranking for total score and the ranking for score on the instructor's questions only.

Submitted by Leonard Feldman, San Jose State University, San Jose, CA 95192.

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Greetings, and welcome to still another Problem Section! This one is something of an anniversary for your Problem Editor, because it has now been one full decade. It was late 1981 that I received the appointment by the late Etta Mae Whitton of Tallahassee Community College, then editor of this journal. I, an eager-beaver problem-solver in the mid-1970s to mid-1980s, suggested the idea to Etta Mae after having joined AMATYC no more than a year earlier (1980). It never occurred to me not to volunteer when I saw that there was no problem section.

Now in 1991, a real problem-solver has recapped the first half of the 1980s' activity in the problem arena. Dr. Stanley Rabinowitz has gathered over 5,000 problems from the period 1980-1984 in two dozen journals, including this one.

I know Stan from our days on the Committee on American Mathematics Competitions. I am always impressed with the energy and dedication he brings to his efforts in problem-solving. His *Index to Math Problems 1980-1984* is, therefore, authoritative. It is available for $50 to individuals. Contact Stan at MathPro Press, P.O. Box 713, Westford, MA 01886.

Midway through this 10-year period I started my own publication, *Recreational & Educational Computing*, which I mention only because it is how I have come to be in contact with Professor Robert Stong, our Solutions Editor since the later 1980s. Bob had initially contacted me—I think it was early 1988—about one of my roundups or columns in *PC Clones*, a short-lived computer publication in the Ziff-Davis stable of computer mags. In turn, Bob became involved with my own publication, and shortly thereafter I invited him to pitch in with AMATYC and serve as Solutions Editor.

As a sidebar, I note that both Etta Mae and Bob have offspring who are talented in mathematics, with Etta Mae's daughter, if memory serves, being in the Georgia State University system. Bob's son, in turn, has solved his share of very tricky problems—you know, the hard *American Mathematical Monthly* kind.

Enough retrospective... To turn to more current matters, I would just like to thank the person who wrote in to ask about Ken Boback's problem last issue, U-3, the one about rationalizing the denominator of a certain fractional expression. The writer commented that it seemed more a teaser than a regular problem.

The answer is that the real problem was the follow-up one, which definitely was not that trivial. But even if the follow-up had not been included already, please remember that the art of problem-solving is not enough; the good problem-solver must become a good problem-poser, one who asks: "What if I changed this value? What if I modified that condition? Is this result still true if...?" Well, you get the idea. That is what real problem-solving thrives on.

Please be reminded that *The AMATYC Review* Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are those challenges and teasers that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematics. Try to avoid the mundane exercises, too.

To submit material for this department, send your **new problem proposals** only, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor at the home address shown below. If you have a solution to your proposal, please include it as well, and/or any relevant comments, history, generalizations, special cases, observations, and/or improvements. Include a mailing label or self-addressed envelope if you'd like a reply.

All solutions to others' proposals should be sent directly to the Solutions Editor at his home address. Addresses are:

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit, PA 18411

Dr. Robert E. Stong
Solutions Editor
The AMATYC Review
150 Bennington Road
Charlottesville, VA 22901

If you see your own problem printed later, please send Professor Stong a copy of your solution, regardless of whether you sent me one originally with your proposal or not. This is not mandatory as policy, but it is the only way to assure that your own solution will be considered for publication or given credit in the list of solvers.

Teaser Solutions

Teasers are math problems that typically take just a few minutes to at most a half-hour to an hour. Solutions follow the next issue. All correspondence to this department should go to the Problem Editor.

Teaser #1. Proposed by the Problem Editor.

A natural number n is perfect if and only if the sum of its divisors that are less than n is n itself. Equivalently, the sum of all its divisors is $2n$. (There are about 30 known perfect numbers, the smallest of which is 6, and the largest of which has 65,087 digits.) Prove that a natural number is perfect if and only if the sum of the reciprocals of all its divisors is 2.

Consider divisors d of n, with $d' = \frac{n}{d}$ the "complementary" divisors. Now,

$$\sum_{d|n}\frac{1}{d} = \sum_{d'|n}\frac{d'}{n} = \frac{1}{n}\sum_{d'|n} d'. \text{ This equals } 2 \Leftrightarrow \sum_{d'|n} d' = 2n.$$

Teaser #2. Proposed by the Problem Editor.

The batting average of a batter is the ratio of hits to times legally at bat. (This does not count walks or other such situations in which the outcome is neither a hit nor an out.) If a player's average is exactly .375, what is the smallest possible number of times the player could have been at bat?

Since $.375 = 3/8$ in lowest terms, the answer is 8.

Comment: Lest anybody poo-poo this teaser, I suggest you offer it to your precalculus, algebra, arithmetic, and other classes – even calculus ones. You may be shocked to find that the vast majority of students in many classes cannot answer this question within five minutes. That such is true *and* some organizations (e.g., NCTM) are calling for calculators even for elementary-school kids does not bode well for our future, I fear. But that is another controversy.

New Teasers

Teaser #3. Two numbers have a sum of 8 and a product of 4. What is the sum of the reciprocals of the two numbers?

Teaser #4. What is the smallest possible value for the sum of a positive real number and its reciprocal? Simple proof, please.

New Problems

Set V Problems are due for ordinary consideration October 1, 1992. However, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Teasers – are always welcome.

An asterisk on a problem indicates that the proposer did not supply a solution with the proposal.

Problem V-1*. Proposed by Travis Thompson, Harding University, Searcy, AR, and edited by the Problem Editor.

Suppose we have a linear array of properties (as in the Parker Brothers' game of Monopoly), said array of length $N + 1$, with property #0 being GO and property #N being JAIL. (This part is not as in Monopoly.) Suppose your token starts at property #1. You toss a fair coin. If heads, you advance your token one to the right; if tails, one to the left.

What is the probability of reaching GO before reaching JAIL? Generalize your answer so that the token starts at property $\#k$ $(0 < k < N)$.

Comment: The proposer has suggested an answer for his Linear Monopoly problem but has not furnished a proof.

Problem V-2. Proposed by the Problem Editor.

You are attempting to complete a collection that has n distinct objects. There are infinitely many copies of each of the n items, and the probability of getting any one of the n types is always $1/n$, regardless of previous draws. On the average, how many items must you draw to complete your collection, which is to say, get at least one of each type? Can you give a nice asymptotic closed-formula expression for this? Generalize to collecting at least any m of the n types $(m \leq n)$.

Problem V-3. Proposed by Mike Dellens, Austin Community College, Austin, TX.

Player X has a .300 lifetime batting average. On the average, how many times theoretically must he come to bat to get one hit? . . . two hits in a row? . . . n hits in a row? (Assume the independence of each at-bat and ignore walks.)

Problem V-4. Proposed by Stephen Plett, Placentia, CA.

Define the cubeness of a rectangular solid (or rectangular parallelepiped) with edges $a, b, c; a \leq b \leq c$; by $Q = a/c$. Call a rectangular solid Pythagorean if its space diagonal and edges a, b, c are all positive integers. Prove that there exist Pythagorean rectangular solids with cubeness arbitrarily close to 1. Are there such solids with cubeness arbitrarily close to 0?

Problem Editor's Comment: A two-dimensional analogue could be posed. One approach to a solution for such a lower-dimensional problem lies in considering the problem of finding (integer-sided) "nearly isosceles" right triangles: those whose legs differ by 1. Two sources that dealt with this question and related ones within the past year (early 1991) are the *Journal of Recreational Mathematics* and *Recreational & Educational Computing*. Mathematically equivalent questions in the latter source involved bowling pins and triangular numbers, all of which variations led to a Pell equation.

Problem V-5. Proposed by Frank Flanigan, San Jose State University, San Jose, CA.

Consider monic real polynomials $P(x)$ with all zeros real as mappings $z \to P(z)$ of the complex z-plane. Which of these mappings compress the y-axis toward the origin in the sense that $|P(i)| < 1$? (Here, $i^2 = -1$ as usual.)

Comments on Problem Set S

A solution to Problem S-2 from Stephen Plett, Fullerton College, Fullerton, CA arrived after the solutions to set S were sent to the publisher. The solution to Problem S-6 is given below. It did not appear in the 1991 Fall issue because of a lack of space.

The Bad Student's Product Rule

S-6. Proposed by Ken Boback and the Problem Editor.

Provide a means of generating examples of pairs of differentiable functions for which the derivative of the product equals the product of the derivatives. In other words, solve $(uv)' = u'v'$ in some general fashion.

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Frank Flanigan, San Jose State University, San Jose, CA; Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada; James Magliano, Union County College, Cranford, NJ; Stephen Plett, Fullerton College, Fullerton, CA; Susan Timar and Carl Timar, Centennial College, Toronto, Ontario, Canada; Joseph Wiener, University of Texas - Pan American, Edinburg, TX; and the proposers.

From $u'v' = u'v + uv'$, division by uv and subsequent manipulation gives, in turn:

$$(u'/u)\,(v'/v) = (u'/u) + (v'/v)$$

$$(\ln u)' = (u'/u) = (v'/v)\,/\,[(v'/v) - 1],$$

$$u = ce^{\int \left(\frac{\frac{v'}{v}}{\frac{v'}{v} - 1} \right) dx}$$

Note. For the special case of $v = \exp(x^2)$, this problem appeared on the Putnam exam. Frank Flanigan noted that this problem occurred as E1661 in the *American Mathematical Monthly,* January 1965. A similar, quite general solution appeared in the October 1989 issue of the *Monthly.*

Set T Solutions

Capture the Flag

T-1. Proposed by Harry J. Smith, Saratoga, CA.

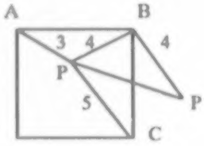
A flat square field contains a flag pole. One of the corners is 3 units away from the pole, the next corner (clockwise) is 4 units away, and the next after that is 5 units away. What is the size of the field?

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Kenneth Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA; Joseph Chance, University of Texas - Pan American, Edinburg, TX; Donald Fuller, Gainesville College, Gainesville, GA; James Magliano, Union County College, Cranford, NJ; John Martin, Santa Rosa Junior College, Santa Rosa, CA; Stephen Plett (2 solutions), Fullerton College, Fullerton, CA; G. Russell, Brevard Community College, Melbourne, FL; Ken Seydel, Skyline College, San Bruno, CA; Problem-Solving Group, Old Rochester Regional High School, Mattapoisett, MA (Dwayne Cameron, Chairman); and the proposer.

Solution 1: Let a be the angle ABP and let b be the angle PBC, with s denoting the side of the square. The law of cosines gives

$$\cos(a) = \frac{s^2 + 4^2 - 3^2}{2 \cdot 4s}$$

$$\cos(b) = \frac{s^2 + 4^2 - 5^2}{2 \cdot 4s}$$



Since $a + b$ is a right angle, $\cos(b) = \sin(a)$, and substituting in the identity $\sin^2(a) + \cos^2(a) = 1$, we have $s^4 - 34s^2 + 65 = 0$. Solving gives $s^2 = 17 + 4\sqrt{14}$.

Solution 2: Rotate the line BP counterclockwise about B through 90 degrees to form the right triangle PBP′. Let c denote the angle CPP′. The law of cosines then gives

$$\cos(c) = \frac{(4\sqrt{2})^2 + 5^2 - 3^2}{2(4\sqrt{2})5} = \frac{3\sqrt{2}}{5}$$

and so $\sin(c) = \sqrt{7}/5$. Applying the law of cosines to the triangle BPC gives

$$\begin{aligned} s^2 &= 4^2 + 5^2 - 2(4)(5)\cos(\pi/4 + c) \\ &= 16 + 25 - 40[(\sqrt{2}/2)(3\sqrt{2}/5) - (\sqrt{2}/2)(\sqrt{7}/5)] \\ &= 17 + 4\sqrt{14}. \end{aligned}$$

Dot Sizes

T-2. Proposed by the Problem Editor.

Let v be an n-tuple of real numbers. Discuss how to permute v to form a new permutation, $p(v)$, so that the dot product of the two vectors is: a) a maximum; b) a minimum.

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Joseph Chance, University of Texas - Pan American, Edinburg, TX; Stephen Plett (2 solutions), Fullerton College, Fullerton, CA; and the proposer.

It follows from

$$v \cdot p(v) = |v|\ |p(v)| \cos(a) = |v|^2 \cos(a) \le |v|^2 = v \cdot v ,$$

where a is the angle between v and $p(v)$, that the maximum product occurs for $p(v) = v$ given by the identity permutation. To find the minimum, it is convenient to permute the entries of v so that

$$v = (v_1, v_2, ..., v_n) \quad \text{with } v_1 \ge v_2 \ge ... \ge v_n.$$

The minimum then occurs for

$$p(v) = (v_n, v_{n-1}, ..., v_2, v_1);$$

i.e., the largest and smallest entries should be paired, etc.

To see this, suppose $a > b$, $c > d$, and consider

$$(..., a, ..., b, ...) \cdot (..., c, ..., d, ...) = ... + ac + ... + bd + ...$$

$$(..., a, ..., b, ...) \cdot (..., d, ..., c, ...) = ... + ad + ... + bc + ...$$

where only corresponding terms are indicated. Now

$$ac + bd - (ad + bc) = (a - b)(c - d) > 0.$$

Thus, switching any two entries in decreasing order will decrease the inner product.

Rolling Rows

T-3. Proposed by the Problem Editor.

For $n \ge 2$, let $w, w^2, ..., w^n = 1$ denote the n nth roots of unity. Form from these an $n \times n$ matrix A by writing this row across, followed by each of the other cyclic permutations below, in order. In each case, the first element in row k is w^k. What is the value of det(A)?

Solved by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA; Joseph Chance, University of Texas - Pan American, Edinburg, TX; Stephen Plett (2 solutions), Fullerton College, Fullerton, CA; and the proposer.

For every k, the kth row is obtained from the first row by multiplying each entry by w^{k-1}. Thus, the rows of A are linearly dependent and det(A) = 0.

Book Reviews

Edited by John J. Edgell, Jr.

ELEMENTARY LINEAR ALGEBRA, by Harvey Gerber, Brooks/Cole, Pacific Grove, California, 1990, ISBN 0-534-11574-8

In the typical undergraduate mathematics major, the course in linear algebra has traditionally been the point of transition between computationally oriented and proof oriented material. With this in mind, there is a serious flaw in this text. Although all of the standard theorems are presented, most proofs are delayed until the end of the section. When the theorems are finally proven, they are not repeated, forcing the reader to flip back to find precisely what is being proven. This tends to belittle the concept of theorem → proof → consequences/applications, at a time when the student should be beginning to "appreciate" it.

While there is a section on numerical analysis, it is listed as optional, and this should be changed. The advances in digital electronics have lead to inexpensive calculators that are capable of performing extensive computations at the touch of a single button. Also, every student of mathematics and science should understand their intrinsic limitations.

With section exercises, chapter quizzes and supplementary exercises, there are enough problems to satisfy the most ravenous of appetites. Each chapter is terminated with a list of keywords and the important theorems and corollaries. Additional optional sections on linear programming, least squares, computer graphics and markov chains provide additional flexibility in the coverage of all the standard linear algebra topics.

Written in a style whose readability is typical of other texts currently available, there is nothing positively unique about this book.

Reviewed by Charles Ashbacher, Kirkwood Community College, Cedar Rapids, IA 52404

A FIRST COURSE IN COMPUTER SCIENCE WITH TURBO PASCAL, Version 4.0, 5.0 and 5.5 by Lowell A. Carmony & Robert L. Holliday. W. H. Freeman & Company, 1991, 570 pp. ISBN 0-7167-8616-2.

This two-color book contains everything one would expect to find in a CS 1 book. It is current, well written, and contains numerous examples and exercises. The table of contents is replete with all CS 1 topics and reflects a traditional approach to teaching problem solving using Turbo Pascal. In addition, the book appears to be error free.

The authors have carefully planned each chapter. This is evidenced by the fact that no topic, example, or exercise, is out of place. Additionally, the words chosen to present the material are perfect. One thing that cannot be said of the material is it is too wordy. In fact, the majority of the chapters in the book represent easy reading, and understanding. Chapter 6, Procedures and Functions, is one of the best I have ever read.

Although several examples scattered throughout the book tend to be a bit long, most are easy to understand and follow. The chapters covering Arrays, 11-13, contain a few long examples. However, I still think they are very effective. I am happy to say that the chapter covering Turbo Pascal is a chapter, and not an appendix. An appendix is reserved for the Turbo Pascal installation procedures. Several design features of the book enhance its effectiveness. In particular, all program comments are displayed in a second color thereby making them easy to identify.

Finally, although the basics of programming the Turbo Pascal are incorporated throughout the book, the reader will be learning problem solving using Turbo Pascal.

Reviewed by Professor Louis Gioia, Nassau Community College, Garden City, NY 11530.

COLLEGE ALGEBRA: A PROBLEM SOLVING APPROACH, 3rd edition, by Walter Fleming and Dale Varberg. Prentice Hall; Englewood Cliffs, New Jersey, 1988, 526pp. ISBN 0-13-141656-1.

Review topics which are usually discussed in one chapter are covered in unusual detail in the first two chapters of this text. This may be a very helpful feature for those students whose mathematics experience is out of date or somewhat inadequate. But the first section, "What is Algebra?" may be unusually intimidating to these same students with poor mathematical experiences. This section may seem overwhelming because of the early emphasis upon multitudes of formulas and many statement type problems. Also, some sections such as 1.2 seem to be either incomplete or perhaps inconsistent. For instance a problem on removing grouping symbols and simplifying expressions, $-2(1+3(6-8)+4)+3$, occurs without a discussion of order of operations. Further, exercises on addition/subtraction of fractions precedes a section on LCM and LCD.

Another concern about the orientation of the text is that processes such as dividing fractions are introduced without seemingly an accompanying explanation of why the process is used or appropriate. Also, describing complex fractions as "four-story fractions" seems to unnecessarily lack in tradition. But referring to the cancellation properties of addition or multiplication as "Rules for Modifying Equations" may be confusing to students in cross referencing to other texts, although it may serve a purpose of expressing the same principle in less intimidating terms.

One good aspect of the book is the diversity of the difficulty of the exercises. Each section begins with standard problems that are straightforward and are meant to develop skills and reinforce the main ideas of the section. After this, the problems become more difficult. These problems provide a challenge for the better students in the class, especially those labeled "teaser problems" at the end of each section.

This text can be effective, but a little reorganization and supplementation by the instructor would enhance its use.

Reviewed by Paul Foutz, Temple Junior College, Temple, TX, 76504.

Send Reviews to: Dr. John J. Edgell, Jr., Editor, Book Reviews, *The AMATYC Review*,
Mathematics Dept., Southwest Texas State University, San Marcos, TX 78666

Software Reviews

Edited by Shao Mah

Title: EXP: The Scientific Word Processor (version 2.0)

Author: Simon L. Smith

Distributor: Brooks/Cole Publishing Company, 511 Forest Lodge Road, Pacific Grove, CA 93950 Phone: (408) 373-0728.

Systems Requirements: IBM compatible computer with at least 384K of RAM, one disk drive (a second floppy drive or a hard drive will make life easier), any graphics card, almost any dot matrix or laser printer, and MS-DOS 2.0 or later.

Price: $295 per copy. Campus and departmental site licenses are available, with price depending on size.

Reviewer's Note: Version 2.1 of EXP has been announced, adding automatically scaling square brackets, support for the VESA Super VGA standard, printer drivers for specific models in the LaserJet family, support for MS-DOS 5.0 and DOS Protected Mode Interface. The price is unchanged. New companion products include a collection of fonts up to 1/2 inch high ($35), a font development kit ($150), and a DVI output driver which will allow you to do typesetting on any device using $\TeX$ fonts ($195).

Introduction

There are three basic approaches to "processing" your mathematical tests, worksheets, articles, or books on a computer. The first is to use a standard word processor. This usually requires some additions by hand, and looks different on the screen than it will on paper, forcing several "print it-fix it" cycles. The results are just so-so, leaving you to hope that your reader, often an inexperienced student, will interpret it the way you had in mind. This method does have one advantage, however; it is cheap. No extra software or hardware is needed.

The second approach is to use "add-in" software. This type of product will work with some word processors to insert boxes of nicely produced mathematical symbols and formulas, usually automatically scaled and aligned. With some, you cannot see both the mathematical material and your regular text at the same time, so you are still guessing what the final product will look like. These add-ins are also not cheap, sometimes costing more than the word processor they work with.

The third approach, used by EXP, is to redesign the word processor as a whole to make the production of mathematical material a part of its normal task. Of course it can produce non-mathematical material as well. The text, symbols, and formulas look the way they are "supposed to" both on the screen and on paper. The nature of such a product is more complex than standard word processors, yet its price is in their range.

Features

EXP has almost all of the features found in the big name word processors except mail merge. Proportionally spaced type is the default, though a change to evenly spaced type (12 to the inch) is accomplished with one key stroke. EXP has a spelling checker (both U.S. and British version), windowing to either different parts of a document or to separate documents, and file import capabilities. Expanded memory can be used when working on large documents.

Switching between the seven fonts provided is easy, and more fonts are available. The default font is 10 point Roman. Also included are 8 point Roman, 9 and 10 point italic, 10 and 17 point sans serif, and the evenly spaced 10 point Roman. Available at a modest extra charge are several 12 point fonts (for all printers) and some larger fonts, up to 36 point size (for laser printers only). Fonts appear correctly on the screen, so you always know what you are using and how it will look. The same is true of the bold and underline features. Print quality is highly hardware dependent. A laser printer will produce pages that look like you just cut them from a book. A 24-pin printer isn't quite that good, but the text it produces under EXP is at least as good as its built-in "letter quality" fonts. I find the output on a 9-pin printer disappointing. It is readable, but nothing more. All printing is done in graphics mode with one pass of the printhead. The two-pass "near letter quality" found in most 9-pin printers is markedly better than EXP output on the same printer. If you are planning to use EXP on a 9-pin printer, I recommend that you get the 12-point fonts; you can then use 10 point fonts for exponents and subscripts and avoid the 8-point fonts which can be hard to read on a 9-pin.

The main reason for using a scientific word processor is to be able to produce the special symbols and formulas. This is where EXP shines. In three years of using EXP I have only failed to find the symbol I wanted once, the "angle" from the mathematics of finance, as in "a angle n at i." Several symbols (fraction bars, brackets, absolute values, radicals, etc.) grow with the expression they contain. Numerators and denominators stay centered automatically, and if you enter an exponent in the denominator, the rest of the term moves down to avoid overprinting the fraction bar. When you type fractions in an equation, the fraction bars stay level with the equal sign. All of this applies to both the screen and the printed page. Thus, EXP is *almost* WYSIWYG (what you see is what you get), actually closer to WYSIWYG than many products which claim to be. The way it falls short is unavoidable. Dot resolution on a monitor screen is not nearly as fine as it is on a printer, making it impossible to display exactly what you "get" on the screen. The screen fonts are sized as closely as resolution will permit to the printer fonts, but not all proportions can be preserved. Most of the time this has no practical effect on the user, but there are two ways in which it can show up. Horizontal position on the screen is only approximate. If one item appears to be exactly under another, it may print off to one side. When horizontal alignment is important, there are two ways to insure it. You may either switch to the evenly spaced font, or else use tabs to locate the items in the essential lines. Vertical position on the screen is also only approximate, but this usually only results in having a little more white space between the lines than you expect.

Two other features deserve mention here. EXP has the capability of importing graphics into your documents. It can handle these files in either the TIFF or PCX format. (Brooks/Cole also sells a graphing package which produces these files.) The other feature is that in EXP it is easy to program macros. A macro is a set of keystrokes or instructions which are then assigned to one keystroke. In fact, this is the essential feature which makes EXP practical to use. The creation of structures like fractions, integrals, and matrices requires the insertion of from several to dozens of codes and characters. A macro accomplishes this in one keystroke, in most cases making knowledge of the special codes involved unnecessary.

EXP comes on both 5.25" and 3.5" disks and must be installed before you can use it. The installation program is easy to follow and does not alter either your autoexec.bat or config.sys files. Graphics cards supported include CGA, EGA, VGA, MCGA, Hercules, AT&T 6300, and a few super VGA cards. Printers listed are Epson (FX, LQ, MX, RX), H.P. (LaserJet, DeskJet Plus), IBM (Graphics, Proprinter, Proprinter X24E), NEC (24-pin), Tandy (24-pin), and Toshiba (24-pin). Almost all printers are compatible with one of these. The program may be installed to either hard or floppy disk (you'll need two if you're using 360K disks).

Base of Learning and Use

With seven or more fonts, hundreds of special symbols, and most available in three sizes, and Greek, German, Gil, and script alphabets (the last three in caps only), it is inevitable that the program will be more complex than an ordinary word processor. A tutorial booklet gives the beginner enough training to type most documents. Rapid and effective use of the product, however, comes after learning a bit about how the program organizes your text and inserts its codes.

The strategy to be employed while using EXP is this: use a macro to produce the basic form you want, and then modify it. Several macros are built-in, anticipating most common needs. (One notable exception is the square root.) For example, to produce the fraction $\frac{3-x}{5}$ you would press F4 (to signal that you want a macro) and then Alt-f (the name of the basic fraction macro); a $\frac{1}{2}$ appears. Move the cursor to the 1, type the $3-x$, and delete the 1. The length of the fraction bar automatically adjusts as you type. Move the cursor to the denominator, type the 5, and delete the 2. You now have the fraction you wanted, without worrying about the codes necessary to put the parts in the right places and draw the bar. For a matrix the procedure is similar. F4 followed by Ctrl-m produces a 3×3 identity matrix. (There are also built-in 2×2 and 4×4 matrices.) Then move the cursor around and change the entries to what you want. Entering special symbols is accomplished by pressing F1 followed by the name of the symbol. For example, F1 *sum 1* produces $\sum$ and F1 *partial* produces ∂.

Eventually most users will want to learn how to make their own structures and macros. This involves learning the "box" concept of EXP. A box is any contiguous set of characters or codes, surrounded by an opening code and a closing code. Depending on which codes are used, the contents of the box might become a sub- or superscript, an upper or lower limit, an entry in a matrix, a roof (placed immediately above the previous box) or a rug, a determinant, a radicand, a numerator, or any of many more possibilities. Creative use of boxes allows you to form virtually any expression you want.

Now the bad news for beginners: EXP uses a command structure to accomplish all these good things. There are no menus, and there is no on-line help. The user must know the commands or else look them up, either on the reference card or in the manual. Fortunately, most of the commands are suggestive of what they do, like the f for fraction and m for matrix in the examples above. But there's no escaping the fact that a great deal must be memorized before you move very quickly. My experience in using EXP and teaching it to others is that the excellent quality of the results is sufficient motivation for an interested person to learn to use the program.

Comments

Although EXP is a very good word processor for both scientific and general use, I do have two gripes. If you don't like the built-in default settings for tabs, margins, line spacing, etc., you must reset them in each document. No direct way of altering the defaults has been provided. After talking with the support person, I was able to devise a way to accomplish this, but it is not for the casual user. I also find the way that the built-in macros are described in both the reference card and the manual to be inadequate. They list the key to press and the set of characters and codes produced. What it will look like is missing, and this is what the user is interested in. If I need to look up how to type a definite integral, I want the reference card to say that to get $\int_0^\infty$, use the ALT-I macro. Instead, the card shows me this:

```
ALT I          F1 int1a rug\0\close roof infinity close  ENTER.
```

Telephone support for registered users of EXP is free, but not toll free. I have not had any trouble getting through, and found the support person to be courteous and helpful. When I had a question she could not answer, she made a suggestion (which did lead to a solution) and offered to contact expert users or the author, if necessary, to find an answer. The company sends out an occasional newsletter-advertisement which contains some tips.

Summary

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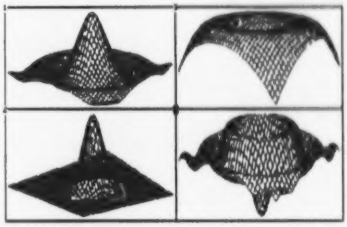

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